《一百一十二學年度第一學期微積分會考答案卷》(A 卷)

第一部份：單選擇題

1	**B**	**2**	**B**	**3**	**C**	**4**	**A**	**5**	**A**
6	**C**	**7**	**D**	**8**	**B**	**9**	**D**	**10**	**D**

第二部份：複選擇題

11	**AC**	**12**	**BCD**	**13**	**AD**	**14**	**BC**	**15**	**ABCD**

第三部份：填空題

1	$(\mathbf{2,1})$	**2**	$\left(\frac{1}{3},\frac{1}{8},-\frac{11}{24}\right)$	**3**	$\left(1,\frac{1}{6}\right)$

一百一十二學年度微積分甲（一）會考計算證明題解答

⦾**Part3：計算/證明題 (Questions of calculations and proofs)**

2. Let $f(x)$ be a differentiable function, and its derivative $f'(x)$ be continuous in $[0,1]$. Let n be a positive integer.

   (1) (3 pts) State the Mean Value Theorem for the function f on the closed interval $[\frac{1}{2n}, \frac{2}{2n}]$.

   (2) (3 pts) Prove that there exists a $c_j \in (\frac{2j-1}{2n}, \frac{2j}{2n})$, $j = 1, 2, \cdots, n$, such that

$$\sum_{k=1}^{2n} (-1)^k f(\frac{k}{2n}) = \frac{1}{2n} \sum_{j=1}^{n} f'(c_j).$$

   (3) (4 pts) Compute

$$\lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^k \, e^{\cos \frac{k}{2n}\pi}.$$

Sample Solution and Marking Policy

(1) If f is continuous on $[\frac{1}{2n}, \frac{2}{2n}]$ and is differentiable on $(\frac{1}{2n}, \frac{2}{2n})$, then there is a number $c \in (\frac{1}{2n}, \frac{2}{2n})$ such that

$$f'(c) = \frac{f(\frac{2}{2n}) - f(\frac{1}{2n})}{\frac{1}{2n}}.$$

Marking Policy

- "If f is continuous on $[\frac{1}{2n}, \frac{2}{2n}]$" is **not necessary** because it is already stated in the question.
- (1 pt) "differentiable on $(\frac{1}{2n}, \frac{2}{2n})$" is needed to state explicitly.
- (1 pt) "there is a number $c \in (\frac{1}{2n}, \frac{2}{2n})$ such that" is needed to state clearly.
- (1 pt) "$f'(c) = \dfrac{f(\frac{2}{2n}) - f(\frac{1}{2n})}{\frac{1}{2n}}$." is necessary to show in the answer.

(2) For each $j = 1, 2, \cdots, n$, by the Mean Value Theorem, there exists $c_j \in (\frac{2j-1}{2n}, \frac{2j}{2n})$ such that

$$f'(c_j) = \frac{f(\frac{2j}{2n}) - f(\frac{2j-1}{2n})}{\frac{1}{2n}}.$$

Therefore, separating even and odd terms, we obtain that

$$\begin{aligned}
\sum_{k=1}^{2n} (-1)^k f(\tfrac{k}{2n}) &= \sum_{j=1}^{n} (-1)^{2j} f(\tfrac{2j}{2n}) + \sum_{j=1}^{n} (-1)^{2j-1} f(\tfrac{2j-1}{2n}) \\
&= \sum_{j=1}^{n} \left(f(\tfrac{2j}{2n}) - f(\tfrac{2j-1}{2n}) \right) \\
&= \frac{1}{2n} \sum_{j=1}^{n} f'(c_j).
\end{aligned}$$

Marking Policy

- "there exists $c_j \in (\frac{2j-1}{2n}, \frac{2j}{2n})$ such that

$$f'(c_j) = \frac{f(\frac{2j}{2n}) - f(\frac{2j-1}{2n})}{\frac{1}{2n}}.\text{''}$$

  is **not necessary** to state.

- (2 pt) "separating even and odd terms, we obtain that

$$\sum_{k=1}^{2n} (-1)^k f(\tfrac{k}{2n}) = \sum_{j=1}^{n} (-1)^{2j} f(\tfrac{2j}{2n}) + \sum_{j=1}^{n} (-1)^{2j-1} f(\tfrac{2j-1}{2n}).\text{''}$$

  is needed to state explicitly.

- (1 pt) "$\sum_{j=1}^{n} \left(f(\frac{2j}{2n}) - f(\frac{2j-1}{2n})\right) = \frac{1}{2n} \sum_{j=1}^{n} f'(c_j)$" is necessary to show in the answer.

(3) We take $f(x) = e^{\cos \pi x}$ which is differentiable and its derivative

$$f'(x) = -\pi \sin \pi x \, e^{\cos \pi x}$$

is continuous in $[0, 1]$. By **(2)**, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^k \, e^{\cos \frac{k}{2n}\pi} &= \lim_{n\to\infty} \frac{1}{2n} \sum_{j=1}^{n} f'(c_j) = \frac{1}{2} \lim_{n\to\infty} \sum_{j=1}^{n} f'(c_j) \, \frac{1}{n} \\
&= \frac{1}{2} \int_0^1 f'(x) \, dx = \frac{1}{2} \left(f(1) - f(0)\right) = \frac{1}{2} \left(e^{-1} - e\right).
\end{aligned}$$

Marking Policy

- (1 pt) "take $f(x) = e^{\cos \pi x}$ which is differentiable and its derivative is continuous in $[0, 1]$" is need to check clearly

- " $f'(x) = -\pi \sin \pi x \, e^{\cos \pi x}$ " is **not necessary** to state explicitly.

- (1 pt) "By (2)

$$\lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^k \, e^{\cos \frac{k}{2n}\pi} = \lim_{n\to\infty} \frac{1}{2n} \sum_{j=1}^{n} f'(c_j).\text{''}$$

  is needed to state explicitly.

- (1 pt) "$\frac{1}{2} \lim_{n\to\infty} \sum_{j=1}^{n} f'(c_j) \, \frac{1}{n} = \frac{1}{2} \int_0^1 f'(x) \, dx$" is necessary to show in the answer. But students do not need to clarify "by Definition of (Riemann) integral".

- (1 pt) $\frac{1}{2} \left(f(1) - f(0)\right) = \frac{1}{2} \left(e^{-1} - e\right)$ is needed to show in the answer but "by fundamental theorem of calculus" is not necessary to clarify