

《一百一十三學年度第一學期微積分會考答案卷》(A 卷)

第一部份：單選擇題

1	D	2	C	3	C	4	B	5	C
6	A	7	B	8	D	9	A	10	A

第二部份：複選擇題

11	ABC	12	ABC	13	AD	14	CD	15	AD
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第三部份：填充題

1	2	
$(-1, \infty)$	A	B
	No or does not exist	e

一百一十三學年度微積分甲 (一) 會考計算證明題評分

3. Let f be a function continuous at 0 and assume that the following limit exists:

$$\lim_{x \rightarrow 0} \left(\frac{f(x)}{x^3} - \frac{3}{x^3} - \frac{2}{x^2} - \frac{1}{x} \right).$$

(A) (2 pts) Find the value of $f(0)$.

(B) (2 pts) Find the value of $f'(0)$.

(C) (4 pts) Assume that $f''(0)$ exists. Find the value of $f''(0)$.

Proof. 令 $g(x) = \frac{f(x)}{x^3} - \frac{3}{x^3} - \frac{2}{x^2} - \frac{1}{x} = \frac{f(x) - 3 - 2x - x^2}{x^3}$, 假設 $\lim_{x \rightarrow 0} g(x) = L$.

(A) (理由, 答案, 各一分.)

Sol1: (說明) 因為 $\lim_{x \rightarrow 0} g(x)$ 極限存在, 且分母 $\lim_{x \rightarrow 0} x^3 = 0$, 所以分子

$$\lim_{x \rightarrow 0} (f(x) - 3 - 2x - x^2) \stackrel{1\text{pt}}{=} 0,$$

$$\text{或(計算)} \lim_{x \rightarrow 0} (f(x) - 3 - 2x - x^2) = \lim_{x \rightarrow 0} x^3 g(x) = 0 \cdot L \stackrel{1\text{pt}}{=} 0.$$

(推導答案)

$$\lim_{x \rightarrow 0} (f(x) - 3 - 2x - x^2) \stackrel{*}{=} f(0) - 3, \implies \underline{f(0) \stackrel{1\text{pt}}{=} 3}.$$

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Sol2: (由定義推導)

$$\begin{aligned} f(0) &\stackrel{*}{=} \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{f(x) - 3 - 2x - x^2}{x^3} x^3 + 3 + 2x + x^2 \\ &= \lim_{x \rightarrow 0} (g(x)x^3 + 3 + 2x + x^2) \stackrel{1\text{pt}}{=} L \cdot 0 + 3 + 2 \cdot 0 + 0^2 \stackrel{1\text{pt}}{=} \underline{3}. \end{aligned}$$

(*: 因為 f 在 0 連續, $\lim_{x \rightarrow 0} f(x) = f(0)$.)

(B) (導數 (不可使用羅畢達法則), 答案, 各一分.)

Sol1: (分開計算)

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{f(x) - 3}{x} - 2 - x \right) &= \lim_{x \rightarrow 0} x^2 g(x) = 0 \cdot L = 0, \\ \lim_{x \rightarrow 0} \left(\frac{f(x) - 3}{x} - 2 - x \right) &\stackrel{(A)}{=} \lim_{x \rightarrow 0} \left(\frac{f(x) - f(0)}{x} - 2 - x \right) \stackrel{1\text{pt}}{=} \underline{f'(0) - 2}, \\ &\implies \underline{f'(0) \stackrel{1\text{pt}}{=} 2}. \end{aligned}$$

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Sol2: (由定義推導)

$$\begin{aligned} f'(0) &\stackrel{1\text{pt}}{=} \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} \stackrel{(A)}{=} \lim_{x \rightarrow 0} \frac{f(x) - 3}{x} \\ &= \lim_{x \rightarrow 0} \left(\frac{f(x) - 3 - 2x - x^2}{x^3} x^2 + 2 + x \right) \\ &= \lim_{x \rightarrow 0} (g(x)x^2 + 2 + x) = L \cdot 0 + 2 + 0 \stackrel{1\text{pt}}{=} 2. \end{aligned}$$

(Attention: No points will be given if L'Hôpital's rule is implemented at the beginning. 注意: 一開始使用 L'Hôpital's rule 就零分. [因為雖然 $f(0) = 3 \implies \lim_{x \rightarrow 0} g(x)$ 是 $\frac{0}{0}$ 未定型, 但 f 未必附近可微分.])

(C) (未定型, 羅畢達法則 (一階導數), 二階導數 (不可使用羅畢達法則), 答案, 各一分.)

Sol1: (分開計算)

$$\lim_{x \rightarrow 0} \frac{f(x) - 3 - 2x}{x^2}$$

is an indeterminate form of $\frac{0}{0}$ (1pt), we may apply L'Hôpital's rule to get

$$(0.1) \quad \lim_{x \rightarrow 0} \frac{f(x) - 3 - 2x}{x^2} \stackrel{1\text{pt}}{=} \lim_{x \rightarrow 0} \frac{f'(x) - 2}{2x} \stackrel{(B)}{=} \frac{1}{2} \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x} \stackrel{1\text{pt}}{=} \frac{f''(0)}{2}.$$

Along with the fact of

$$0 = 0 \cdot L = \lim_{x \rightarrow 0} xg(x) = \lim_{x \rightarrow 0} \frac{f(x) - 3 - 2x}{x^2} - 1,$$

this implies $f''(0) = 2$ (1pt)

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Sol2: (合併計算)

$$(A) \quad f(0) = 3 \implies \lim_{x \rightarrow 0} \frac{f(x) - 3 - 2x}{x^2}$$

is an indeterminate form (未定型) of type $\frac{0}{0}$, 1pt

$$\begin{aligned} 0 &= 0 \cdot L = \lim_{x \rightarrow 0} xg(x) = \lim_{x \rightarrow 0} \left(\frac{f(x) - 3 - 2x}{x^2} - 1 \right) \\ &\stackrel{1\text{pt}}{=} \lim_{x \rightarrow 0} \left(\frac{f'(x) - 2}{2x} - 1 \right) \stackrel{(B)}{=} \lim_{x \rightarrow 0} \left(\frac{f'(x) - f'(0)}{2x} - 1 \right) \stackrel{1\text{pt}}{=} \frac{f''(0)}{2} - 1, \\ &\implies f''(0) \stackrel{1\text{pt}}{=} 2. \end{aligned}$$

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(Attention: The reasoning of the last equation in (0.1) can not be replaced by L'Hôpital's rule.) (注意: L'H 只能用在第一次, 第二次不可以使用. [因為 f' 未必附近可微分])