一百一十四學年度微積分甲 (一) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 9 頁。

(3) 試卷包括選擇題、填充題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Let k be a real number. Consider the function

$$f(x) = \begin{cases} \dfrac{1-\cos(kx)}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Find the value of $f'(0)$.

(A) 0 (B) $\dfrac{k}{2}$ (C) $\dfrac{k^2}{2}$ (D) Does not exist

2. Let $a > 0$ and define

$$F(x) = e^x + \frac{1}{5}x^5 - \frac{2\sqrt{a}}{3}x^3 + ax, \qquad x \in (-\infty, \infty).$$

How many real roots does the equation $F(x) = 0$ have on $(-\infty, \infty)$?

(A) 0 (B) 1 (C) 2 (D) 3

3. Let g be differentiable on $(-\infty, \infty)$. Assume $g(0) = 1$ and that $g(x)$ satisfies

$$(g(x))^2 + (x^2 + 3x + 2)g(x) - 3 = 0 \quad \text{for all real numbers } x.$$

Find the value of $g'(0)$.

(A) -3 (B) $-\dfrac{3}{4}$ (C) $\dfrac{3}{4}$ (D) 3

4. Find the number of inflection points of the curve $y = f(x) = \dfrac{e^x}{1+x}$.

(A) 0 (B) 1 (C) 2 (D) 3

5. Find the value of the integral

$$I = \int_0^1 \left(xe^{2x} + \sec\left(\frac{\pi}{4}x\right) \right) \, dx.$$

(A) $\dfrac{e^2+1}{4} + \dfrac{4}{\pi}\ln(\sqrt{2}+1)$

(B) $\dfrac{e^2-1}{4} + \dfrac{4}{\pi}\ln(\sqrt{2}-1)$

(C) $\dfrac{e^2+1}{4} + \dfrac{8}{\pi}\ln(\sqrt{2}+1)$

(D) $\dfrac{e^2-1}{4} + \dfrac{8}{\pi}\ln(\sqrt{2}-1)$

6. Let f be a differentiable function with $\lim\limits_{x\to 6} f(x) = 0$ and $\lim\limits_{x\to 6} f'(x) = 1$. Find the value of

$$\lim_{x\to 6} \frac{\displaystyle\int_6^x \left(6t \int_t^6 f(u)\, du \right) dt}{(6-x)^3}.$$

(A) 2 (B) 4 (C) 6 (D) 8

7. Find the value of

$$\lim_{n\to\infty} \left\{ \frac{\cos\frac{\pi}{2n}}{\pi - n} + \frac{\cos\frac{2\pi}{2n}}{\pi - n} + \cdots + \frac{\cos\frac{\pi}{2}}{\pi - n} \right\}.$$

(A) $\dfrac{2}{\pi}$ (B) π (C) $-\pi$ (D) $-\dfrac{2}{\pi}$

8. Suppose that there is a line through the origin that divides the region bounded by the parabola $y = 2x - 4x^2$ and the x-axis into two regions with equal area. What is the slope of the line?

(A) $2 - \frac{1}{\sqrt[3]{2}}$ (B) $2 - \sqrt[3]{2}$ (C) $2 - \sqrt[3]{3}$ (D) $2 - \sqrt[3]{4}$

9. Consider the function

$$y = f(x) = e^{x^2}, \qquad 0 \le x \le 1.$$

- Let I_1 be the volume of the solid obtained by rotating the region bounded by the graph of $y = f(x)$, the x-axis, and the vertical lines $x = 0$ and $x = 1$, about the vertical line $x = -6$.
- Let I_2 be the area of the surface obtained by rotating the curve $y = f(x)$, $0 \le x \le 1$, about the x-axis.
- Let I_3 be the arc length of the curve $y = f(x)$, $0 \le x \le 1$.

Which of the following is **TRUE**?

(A) $I_1 > I_2 > I_3$ (B) $I_2 > I_1 > I_3$ (C) $I_1 > I_3 > I_2$ (D) $I_2 > I_3 > I_1$

10. Let a and b be positive integers with $1 \le a \le 3$ and $1 \le b \le 5$. How many ordered pairs (a, b) are there for which the improper integral

$$\int_0^{\pi/2} \frac{(1 - \cos x)^a}{x^b}\, dx$$

converges?

(A) 9 (B) 10 (C) 11 (D) 12

◎ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points, for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Which of the following are **TRUE**?

    (A) Every continuous function defined on (a, b) has antiderivatives on (a, b).

    (B) Every continuous function defined on (a, b) attains its absolute minimum.

    (C) If $f'(x) < 0$ for all real numbers x, then $\lim_{x\to\infty} f(x) = -\infty$.

    (D) If $f(0) = 0$ and the derivative f' is increasing on $(0, \infty)$, then $\dfrac{f(x)}{x}$ is increasing on $(0, \infty)$.

12. Let $p > 0$ be a fixed constant and define

$$f(x) = \lim_{n\to\infty} \frac{x^{n+p} - x^{-n}}{x^n + x^{-n}}, \qquad x > 0.$$

    Which of the following are **TRUE**?

    (A) $|\lim_{x\to 1^+} f(x) - \lim_{x\to 1^-} f(x)| = p$ for all real numbers $p \geq 2$.

    (B) If $x > 1$, then $f(x) = x^p$.

    (C) If $0 < x < 1$, then $f(x) = -1$.

    (D) $\lim_{x\to 0^+} f(x)$ exists.

13. Let

$$f(x) = \tan^{-1}(g(x)),$$

where $g(x)$ is differentiable on an open interval I. Which of the following are **TRUE**?

(A) The range of f is always contained in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(B) If $g'(x) > 0$ for all $x \in I$, then $f(x)$ is increasing on I.

(C) If $g(x) = e^x \cos x$, then $f'(0) = -\frac{1}{2}$.

(D) If there exists a real number M such that $|g'(x)| \leq M$ for all $x \in I$, then

$$|f'(x)| \leq M \quad \text{for all } x \in I.$$

14. Let $f(x)$ be the function defined by

$$f(x) = \frac{\tan(\pi x)}{\sqrt{x(x-2)}}.$$

Which of the following statements are **TRUE**?

(A) The line $y = 0$ is a horizontal asymptote of the graph of f.

(B) The line $x = 2$ is a vertical asymptote of the graph of f.

(C) The line $x = -\frac{1}{2}$ is a vertical asymptote of the graph of f.

(D) The line $x = \frac{5}{2}$ is a vertical asymptote of the graph of f.

15. Consider the curve C parametrized by

$$x = a(1 + \cos t)\sin t, \qquad y = a(1 + \cos t)\cos t, \qquad 0 \leq t < 2\pi,$$

where $a > 0$ is a constant. Which of the following statements are **TRUE**?

(A) C is symmetric with respect to the y-axis.

(B) There are exactly three points on C where the tangent lines are horizontal.

(C) The arc length of C is $\int_0^{2\pi} a\sqrt{2(1 + \cos t)}\, dt$.

(D) The area enclosed by C is $\frac{3}{2}\pi a^2$.

◎ **Part 3: 填充題 (Fill-in-the-Blank Questions)**
(一個題組，共十分。) (There are one question sets worth a total of 10 points.)

1. Assume that the parabola

$$y = f(x) = ax^2 + bx + 2025 \ln c$$

passes through the origin, and that $f(x) \geq 0$ when $0 \leq x \leq 1$, where a, b and c are constants.

(A) (5 pts) Let V be the volume of the solid obtained by rotating the region bounded by curves

$$x = 0, \quad x = 1, \quad y = 0, \quad y = f(x),$$

about the x-axis. Find the value of the volume V (in terms of a, b and c).

(B) (5 pts) Suppose that the area of the region bounded by curves $x = 0$, $x = 1$, $y = 0$ and $y = f(x)$ is $\dfrac{1}{3}$. Find a, b and c such that V attains an absolute minimum value.

◎ **Part 4: 計算/證明題 (Calculation/Proof Questions)**
(一個題組，共十五分。) (There is one question set worth a total of 15 points.)
(計算/證明題答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分；如果計算錯誤，則酌給部分分數；如果只有答案對，但觀念錯誤，或是過程不合理，則無法得到分數。)
(Answer the question of calculations and proofs as thoroughly as possible. In the case of computational errors, partial credit may be given. However, if only the answer is correct but there are conceptual errors or an unreasonable process, no credit will be awarded.)

1. Consider the integral

$$I = \int_0^\infty \frac{u^2}{(1+u^2)(1+2u^2)}\, du.$$

(A) (2 pts) Consider

$$\frac{u^2}{(1+u^2)(1+2u^2)} = \frac{\alpha}{1+u^2} + \frac{\beta}{1+2u^2},$$

where α and β are constants. Find α and β.

(B) (5 pts) Evaluate the value of I.

(C) (8 pts) Evaluate the value of the integral

$$\int_0^{1/2} \frac{x^2}{(1+x^2)\sqrt{1-x^2}}\, dx.$$