《一百一十四學年度第一學期微積分會考答案卷》(A 卷)

第一部份：單選擇題

1	**C**	**2**	**B**	**3**	**B**	**4**	**A**	**5**	**A**
6	**C**	**7**	**D**	**8**	**D**	**9**	**A**	**10**	**C**

第二部份：複選擇題

11	**AD**	**12**	**BCD**	**13**	**ABD**	**14**	**CD**	**15**	**ABCD**

《一百一十四學年度第一學期微積分會考答案卷》(B 卷)

第一部份：單選擇題

1	**B**	**2**	**C**	**3**	**A**	**4**	**A**	**5**	**B**
6	**D**	**7**	**C**	**8**	**A**	**9**	**D**	**10**	**C**

第二部份：複選擇題

11	**AD**	**12**	**CD**	**13**	**ABD**	**14**	**BCD**	**15**	**ABCD**

第三部份：填充題（A、B 卷相同）

1	
(A)	**(B)**
$\pi(\frac{1}{5}a^2+\frac{1}{2}ab+\frac{1}{3}b^2)$ **or** $\pi(\frac{1}{5}a^2+\frac{1}{2}ab+\frac{1}{3}b^2+1350a\ln c$ $+2025b\ln c+(2025\ln c)^2)$	$(a,b,c)=(-\frac{5}{4},\frac{3}{2},1)$

一百一十四學年度微積分甲 (一) 會考計算題

1. Consider the integral
$$I = \int_0^\infty \frac{u^2}{(1+u^2)(1+2u^2)}\,du.$$

   (a) (2 pts) Consider
$$\frac{u^2}{(1+u^2)(1+2u^2)} = \frac{\alpha}{1+u^2} + \frac{\beta}{1+2u^2}, \tag{1}$$
   where α and β are constants. Find α and β.

   (b) (5 pts) Evaluate the value of I.

   (c) (8 pts) Evaluate the value of the integral
$$\int_0^{1/2} \frac{x^2}{(1+x^2)\sqrt{1-x^2}}\,dx.$$

Solution: (a) By (1), we have
$$\frac{u^2}{(1+u^2)(1+2u^2)} = \frac{\alpha(1+2u^2)+\beta(1+u^2)}{(1+u^2)(1+2u^2)}. \tag{2}$$
Then $u^2 = (\alpha+\beta) + (2\alpha+\beta)u^2$. It follows that
$$\alpha+\beta = 0, \qquad 2\alpha+\beta = 1.$$
We further obtain
$$\alpha = 1, \quad \beta = -1.$$
註：也可以將(2)兩邊同乘 $(1+u^2)$，並令 $u^2=-1$ 得到 $\alpha=1$；兩邊同乘 $(1+2u^2)$ 並令 $u^2=-1/2$ 得到 $\beta=-1$.

(b) By (a), we see that
$$I = \int_0^\infty \left(\frac{1}{1+u^2} - \frac{1}{1+2u^2}\right) du = \int_0^\infty \frac{1}{1+u^2}\,du - \int_0^\infty \frac{1}{1+2u^2}\,du.$$
The first integral in the above is
$$\int_0^\infty \frac{1}{1+u^2}\,du = \lim_{t\to\infty} \left[\tan^{-1} u\right]_0^t = \frac{\pi}{2}.$$
For the second integral, we set $v=\sqrt{2}\,u$, $dv = \sqrt{2}\,du$ and obtain
$$\int_0^\infty \frac{1}{1+2u^2}\,du = \frac{1}{\sqrt{2}}\int_0^\infty \frac{1}{1+v^2}\,dv = \frac{\pi}{2\sqrt{2}}.$$
Therefore,
$$I = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right) \quad \text{or} \quad \frac{\pi}{4}(2-\sqrt{2}).$$

(c) Let $x = \sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = \cos\theta d\theta$ and when $x = 0$, $\theta = 0$；when $x = 1/2$, $\theta = \pi/6$. Then we have

$$J := \int_0^{\pi/6} \frac{\sin^2\theta}{(1+\sin^2\theta)\cos\theta}\cos\theta d\theta = \int_0^{\pi/6} \frac{\sin^2\theta}{1+\sin^2\theta} d\theta \tag{3}$$

Set $u = \tan\theta$. Then

$$du = \sec^2\theta\, d\theta = (1+\tan^2\theta)\, d\theta = (1+u^2)\, d\theta \quad\Longrightarrow\quad d\theta = \frac{du}{1+u^2}.$$

By some computations,

$$\frac{\sin^2\theta}{1+\sin^2\theta} = \frac{\frac{u^2}{1+u^2}}{\frac{1+2u^2}{1+u^2}} = \frac{u^2}{1+2u^2}.$$

Also, when $\theta = 0$, $u = \tan 0 = 0$; when $\theta = \pi/6$, $u = \tan(\pi/6) = 1/\sqrt{3}$. So

$$J = \int_0^{1/\sqrt{3}} \frac{u^2}{(1+u^2)(1+2u^2)}\, du. \tag{4}$$

Using the decomposition from part (a),

$$J = \int_0^{1/\sqrt{3}} \frac{u^2}{(1+u^2)(1+2u^2)}\, du = \int_0^{1/\sqrt{3}} \frac{1}{1+u^2}\, du - \int_0^{1/\sqrt{3}} \frac{1}{1+2u^2}\, du$$

Compute each term:

$$\int_0^{1/\sqrt{3}} \frac{1}{1+u^2}\, du = \left[\tan^{-1} u\right]_0^{1/\sqrt{3}} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

and set $v = \sqrt{2}\,u$,

$$\int_0^{1/\sqrt{3}} \frac{1}{1+2u^2}\, du = \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}/\sqrt{3}} \frac{1}{1+v^2}\, dv = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{6}}{3}\right).$$

So we have

$$\int_0^{1/2} \frac{x^2}{(1+x^2)\sqrt{1-x^2}}\, dx = \frac{\pi}{6} - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{6}}{3}\right).$$

註 2：解題過程中不一定會出現(4), 例如再前面推導時

$$J := \int_0^{\pi/6} \frac{\sin^2\theta}{1+\sin^2\theta} d\theta = \int_0^{\pi/6} \frac{\sin^2\theta + 1 - 1}{1+\sin^2\theta} d\theta = \int_0^{\pi/6} \left(1 - \frac{1}{1+\sin^2\theta}\right) d\theta$$
$$= \frac{\pi}{6} - \int_0^{\pi/6} \frac{1}{1+\sin^2\theta} d\theta.$$

則對應於(4)會變成

$$J = \frac{\pi}{6} - \int_0^{1/\sqrt{3}} \frac{1}{1+2u^2}\, du.$$