一百一十四學年度微積分甲 (二) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 8 頁。

(3) 試卷包括選擇題、填充題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Let D be the region enclosed by the right loop of the polar curve

$$r^2 = 2\cos 2\theta, \qquad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}.$$

Find the area of D.

(A) $\frac{1}{2}$; (B) 1; (C) 2; (D) 4.

2. Let z be a differentiable function of u and v, and let u and v be differentiable functions of x and y. Suppose that at (x_0, y_0) we have

$$\frac{\partial z}{\partial u} = 3, \qquad \frac{\partial z}{\partial v} = -1, \qquad \frac{\partial u}{\partial x} = 2, \qquad \frac{\partial u}{\partial y} = 1, \qquad \frac{\partial v}{\partial x} = -2, \qquad \frac{\partial z}{\partial y} = 7.$$

Evaluate $\frac{\partial v}{\partial y}$ at (x_0, y_0).

(A) -4; (B) -2; (C) 2; (D) 4.

3. Let $f(x, y)$ be a function differentiable at $(1, 2)$. If the tangent plane to the surface $z = f(x, y)$ at $(1, 2, 3)$ is $2x - y + z = 3$, then $f(1, 2) + f_x(1, 2) + f_y(1, 2)$ equals

(A) -1; (B) 0; (C) 1; (D) 2.

4. The interval of convergence of the power series

$$\sum_{n=1}^{\infty}\left(\frac{4^n}{\sqrt{n}}+n\right)x^{2n}$$

equals

(A) $(-\frac{1}{2},\frac{1}{2})$; (B) $(-1,1)$; (C) $(-\frac{1}{2},\frac{1}{2}]$; (D) $[-1,1]$.

5. Let $f(x,y)$ be a function differentiable at (x_0,y_0) and set

$$u=\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\rangle,\quad v=\langle\frac{1}{2},\frac{\sqrt{3}}{2}\rangle,\quad w=\langle-\frac{\sqrt{3}}{2},-\frac{1}{2}\rangle.$$

If $D_uf(x_0,y_0)=\sqrt{2}$ and $D_vf(x_0,y_0)=\sqrt{3}$, then $D_wf(x_0,y_0)$ equals

(A) -1; (B) $\sqrt{2}$; (C) $-\sqrt{3}$; (D) $\sqrt{6}$.

6. Let $D=[0,1]\times[0,1]$ and f be a function defined by

$$f(x,y)=\sum_{n=1}^{\infty}\frac{(xy)^n y}{n!}.$$

Then, $\iint_D f(x,y)dA$ equals

(A) $e-\dfrac{1}{2}$; (B) $e-\dfrac{3}{2}$; (C) $e-2$; (D) $e-\dfrac{5}{2}$.

7. Let E be the solid region between the spheres

$$x^2+y^2+z^2=4 \qquad \text{and} \qquad x^2+y^2+z^2=9.$$

Find the value of

$$\iiint_E (x^2+y^2+z^2)\,dV.$$

(A) $\dfrac{211\pi}{5}$; (B) $\dfrac{422\pi}{5}$; (C) $\dfrac{844\pi}{5}$; (D) $\dfrac{1688\pi}{5}$.

8. Let D be the parallelogram bounded by the lines

$$x+y=0,\qquad x+y=2,\qquad x-y=0,\qquad x-y=1.$$

Then

$$\iint_D e^{x+y}\cos(x-y)\,dA$$

is equal to

(A) $(e^2-1)\sin 1$; (B) $\dfrac{1}{2}(e^2-1)\sin 1$; (C) $(e-1)\sin 1$; (D) $\dfrac{1}{2}(e^2+1)\sin 1$.

9. Let E be the solid inside both the cube $\{(x,y,z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$ and the ball $\{(x,y,z) : x^2+y^2+z^2 \leq 2\}$. Then, the surface area of E is $8\pi - 6\times a + 6\pi$, where a equals

(A) $(2-\sqrt{2})\pi$; (B) $(2-2\sqrt{2})\pi$; (C) $(4-2\sqrt{2})\pi$; (D) $(4-4\sqrt{2})\pi$.

10. Let
$$a_n = \ln\left(1+\frac{1}{n}\right).$$
Which of the following statements is true?

(A) $\lim\limits_{n\to\infty} na_n$ does not exist.

(B) $\sum\limits_{n=1}^{\infty} a_n$ converges.

(C) $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$ diverges.

(D) $\sum\limits_{n=1}^{\infty} a_n^2$ converges.

◎ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points, for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Under which of the following constraints does $f(x,y,z) = x^2 - y^2 + z^2$ achieve its absolute minimum?

(A) $x^4+y^4+z^4 = 1$; (B) $x^2-y^2 = 1$; (C) $x^2+y^2-z^2 = 1$; (D) $x+y+z = 0$.

12. Which of the following series are convergent?

(A) $\sum\limits_{n=1}^{\infty}(-1)^n \ln(2+\frac{1}{n})$; (B) $\sum\limits_{n=1}^{\infty}\frac{\cos n}{n^2+1}$; (C) $\sum\limits_{n=1}^{\infty} n^{-n}$; (D) $\sum\limits_{n=1}^{\infty}\sin n$.

13. Let

$$D = \{(x,y) : x^2 + y^2 \le 1\}.$$

Which of the following statements are true?

(A) $\displaystyle\iint_D xe^{x^2+y^2}\, dA = 0.$

(B) $\displaystyle\iint_D (x^2 - y^2)\, dA = 0.$

(C) $\displaystyle\iint_D xy\, dA > 0.$

(D) $\displaystyle\iint_D (x+y)^2\, dA = \iint_D (x-y)^2\, dA.$

14. Consider

$$f(x,y,z) = xyz,$$

subject to the constraint

$$g(x,y,z) = x^2 + y^2 + z^2 - 3 = 0.$$

To find the extreme values of f under this constraint, we use the method of Lagrange multipliers. That is, find all values of x, y, z, and λ such that

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z), \\ g(x,y,z) = 0. \end{cases} \tag{1}$$

Which of the following statements are true?

(A) If f attains its absolute maximum at (x_0, y_0, z_0), then $x_0^2 = y_0^2 = z_0^2$.

(B) If (x, y, z, λ) is a solution of (1), then $\lambda = \pm\dfrac{1}{2}$.

(C) The maximum value of f under the constraint is 1.

(D) The maximum value of f under the constraint is attained at exactly two points.

15. Define

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Which of the following statements are true?

(A) f is continuous at $(0,0)$.

(B) $D_{\mathbf{u}}f(0,0)$ exists for every unit vector $\mathbf{u}$.

(C) $D_{\mathbf{u}}f(0,0) = \nabla f(0,0) \cdot \mathbf{u}$ for every unit vector $\mathbf{u}$.

(D) f is differentiable at $(0,0)$.

◎ **Part 3: 填充題與計算/證明題 (Fill-in-the-Bank Questions, and Questions of calculations and proofs)**

(兩個題組，共二十五分。第一題為填充題，第二題為計算證明題。)

(There are two questions worth a total of 25 points. The first question is a fill-in-the-blank question, and the second one is a question of calculations and proofs.)

(計算/證明題答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分；如果計算錯誤，則酌給部分分數；如果只有答案對，但觀念錯誤，或是過程不合理，則無法得到分數。)

(Answer the question of calculations and proofs as thoroughly as possible. In the case of computational errors, partial credit may be given. However, if only the answer is correct but there are conceptual errors or an unreasonable process, no credit will be awarded.)

1. Suppose

$$f(x) = \frac{\ln(1+x)}{x}, \quad x \neq 0,$$

where $f(0)$ is defined such that f is continuous at 0. Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (|x| < 1).$$

(a) (5 points) Find a_n for $n = 0, 1, 2, \ldots$.

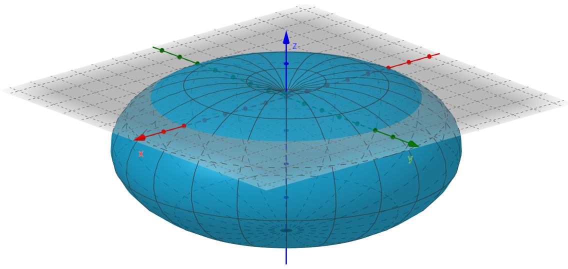
(b) (5 points) Find the interval of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

(c) (5 points) Evaluate the integral $\int_0^1 f(x)dx$ (Hint: you may use the fact that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$).

2. Let S be the surface whose equation in spherical coordinates is given by

$$\rho = 1 - \cos\phi,$$

where ϕ is measured from the positive z-axis. Let E be the solid enclosed by S and let D be the cross-section of E by the vertical half-plane $\theta = \pi/3$.



(a) (5 points) Find the volume of E.

(b) (5 points) Find the area of D.