一百零八學年度第一學期微積分會考試題 (A 卷)

說明 ：

(1) 答題之前請先檢查所取得之試卷與答案卷是否正確。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 7 頁。

(3) 試卷包括選擇題與計算/證明，總分共計 100 分，占學期成績之 30%。考卷成績將做為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part1：單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. The limit $\lim_{x\to 0}(\cos x - \sin x)^{\frac{1}{\tan x}}$ is

   (A) 0. (B) 1. (C) e. (D) e^{-1}.

2. Let R be the region enclosed by $y^2 - y + x = 0$ and $x = 0$. The volume of the solid obtained by rotating R about the line $y = 2$ is

   (A) $\pi/4$. (B) $\pi/2$. (C) π. (D) $3\pi/2$.

3. Consider the following functions.

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}, \qquad g(x) = \begin{cases} x\cos x, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

   Which of the following statements is **TRUE**?

   (A) $\lim_{x\to 0} f(x) = 1$. (B) $\lim_{x\to 0} g(x) = 1$.
   (C) $\lim_{x\to 0} g(f(x)) = \cos 1$. (D) $\lim_{x\to 0} f(g(x)) = 0$.

4. The length of the curve $y = \ln(\cos x)$ from $(0, 0)$ to $(\pi/4, \ln(1/\sqrt{2}))$ is

   (A) $\ln(\sqrt{2}+1)$. (B) $\ln(2\sqrt{2}+1)$. (C) $\ln(\frac{1}{\sqrt{2}}+1)$. (D) 1.

5. Let f be an odd function defined on $(-\infty, \infty)$. If f' is continuous and $f(1) = 1$, then $\int_0^2 xf'(1-x)dx$ equals

   (A) 1. (B) -1. (C) 2. (D) -2.

6. Let f be a differentiable function defined on $(0, \infty)$. Which of the following must be **TRUE**?

   (A) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} f'(x) > 0$.

   (B) If $\lim_{x\to\infty} f'(x) > 0$, then $\lim_{x\to\infty} f(x) = \infty$.

   (C) If $\lim_{x\to 0^+} f(x) = \infty$, then $\lim_{x\to 0^+} f'(x) = -\infty$.

   (D) If $\lim_{x\to 0^+} f'(x) = -\infty$, then $\lim_{x\to 0^+} f(x) = \infty$.

7. Let $V(t)$ be the volume obtained by rotating the region

$$A(t) := \{(x, y) : 0 \le x \le t, 0 \le y \le \frac{1 + \sin^2(x)}{2}\}$$

   about the y-axis. For what value of $-\infty < r < \infty$ and $0 < c < \infty$ does one have

$$\lim_{t\to 0^+} \frac{V(t)}{t^r} = c?$$

   (A) $r = 1$, $c = \pi$. (B) $r = 1$, $c = \pi/2$.
   (C) $r = 2$, $c = \pi$. (D) $r = 2$, $c = \pi/2$.

8. Assume that f is continuous and satisfies the following equation

$$\lim_{n\to\infty} \frac{f(\frac{x^2}{n}) + f(\frac{2x^2}{n}) + \cdots + f(\frac{(n-1)x^2}{n}) + f(\frac{nx^2}{n})}{n} = \frac{\sin(\pi x)}{x},$$

   for any real number $x \neq 0$. Find $f(1)$.

   (A) π. (B) $\pi/2$. (C) 0. (D) $-\pi/2$.

9. Find the condition under which the value of the following integral must be positive

$$\int_0^{2\pi/\beta} e^{\alpha t}\Big(\alpha\cos(\beta t) - \beta\sin(\beta t)\Big)dt.$$

   (A) $\alpha > 0, \beta < 0$. (B) $\beta > 2\pi\alpha$. (C) $\alpha\beta > 0$. (D) $\beta < 2\pi\alpha$.

10. Consider the function $F(x) = (f \circ g')(x)$, where

$$g(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 1 \\ -\frac{2}{x} - \frac{x-1}{2} & \text{if } x > 1 \end{cases}, \ f(x) = 4x^3 - 15x^2 + 12x.$$

Which of the following about the absolute maximum value y_M and absolute minimum value y_m of F on $\{x : 0 \leq x \leq 2 \text{ and } F(x) \text{ is defined}\}$ is true ?

(A) $y_M = 1, y_m = 0$.

(B) $y_M = \frac{11}{4}, y_m = -4$.

(C) $y_M = \frac{11}{4}$, no absolute minimum value.

(D) $y_M = 1, y_m = -4$.

◎ Part2：多選擇題
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。答錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points , AB => 3 points, AD => 0 points, ABD => 0 points

11. Which of the following statements must be **TRUE**?

(A) If $\lim_{x\to 0} |f(x)| = |L|$, then $\lim_{x\to 0} f(x) = L$.

(B) If $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} |f(x)| = |L|$.

(C) If f is an odd function defined in $(-\infty, \infty)$, then $\lim_{x\to 0} f(x) = 0$.

(D) Let f and g be defined in $(-\infty, \infty)$. If f is an even function and g is an odd function, then both $f \circ g$ and $g \circ f$ are even functions.

12. Consider the function $f(x) = \int_0^x |\sin(t)|\, dt$, where $-\infty < x < \infty$. Which of the following statements are **TRUE**?

    (A) $f(x)$ is an increasing function.
    (B) $f(x)$ is a differentiable function.
    (C) $f'(x)$ is a continuous function.
    (D) $f'(x)$ is a differentiable function.

13. Define $f(x) = \int_1^{x^2} \frac{t^2}{2(t^2+1)} dt$. Which of the following statements are **TRUE**?

    (A) f is continuous on $(-\infty, \infty)$.

    (B) f' has a slant asymptote $y = x$.

    (C) f is concave upward on $(-\infty, \infty)$.

    (D) The graph of f' is symmetric about the origin.

14. Consider the following function:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

    Which of the following statements are **TRUE**?

    (A) f is differentiable at 0.

    (B) f has infinitely many local maxima.

    (C) The curve $y = f(x)$ has infinitely many inflection points.

    (D) The line $y = x$ is a slant asymptote of $y = f(x)$.

15. Let f and g be functions which are continuous on $[a, b]$ and differentiable on (a, b) where $a < b$. Which of the following conditions can guarantee that f and g are equal on $[a, b]$?

    (A) $\lim_{y \to x} f(y) = \lim_{y \to x} g(y)$ for any x on (a, b).
    (B) $f'(x) = g'(x)$ for any x on (a, b).
    (C) $\int_a^x f(t)\, dt = \int_a^x g(s)\, ds$ for any x on (a, b).
    (D) $f(x) + \int_a^x f(t)\, dt = g(x) + \int_a^x g(s)\, ds$ for any x on (a, b).

◎**Part3：計算/證明題 (Questions of calculations and proofs)**
(答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分。如果計算錯誤，則酌給部分分數。如果只有答案對，但觀念錯誤，或是過程不合理，則無法得到分數。)
(Answer the problems as thoroughly as possible. Be sure to include all your work. Partial credit will be given even if the answer is not fully correct.)

1. A curve C is defined by the equation $y^2 = x^2(x+2)$. From Figure 1, you can see that part of the curve forms a loop.

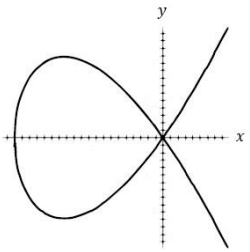

Figure1. The graph of C

(A) (5 points) At what points does the curve C have horizontal tangents?

(B) (5 points) Find the area of the region enclosed by the loop.

2. Consider the parametric curve $\gamma(t) = (e^{-t}\sin t, e^{-t}\cos t)$.

(A) (4 points) Compute the length of $\{\gamma(t)|0 \le t < \infty\}$.

(B) (4 points) Determine whether the area of the surface obtained by rotating $\{\gamma(t)|0 \le t < \infty\}$ about $x = 1$ is finite or infinite.

(C) (7 points) For $i = 1, 2, 3, 4$, let L_i be the tangent line to γ at $\gamma(\pi i/2)$. Find the area of the region enclosed by L_1, L_2, L_3, L_4.