一百零九學年度第一學期微積分會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 7 頁。

(3) 試卷包括選擇題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將做為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part1：單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. The limit $\lim_{x\to 3}\int_3^x \frac{x\sin(t)}{t(x-3)}\,dt$ is

   (A) $\sin 3$; (B) 0; (C) 3; (D) It does not exist.

2. The **derivative** of the function $f(x)=\left(\ln(1+x^2)\right)^x$ is

   (A) $\frac{2x\left(\ln(1+x^2)\right)^{x-1}}{1+x^2}$;

   (B) $\frac{2x^2\left(\ln(1+x^2)\right)^{x-1}}{1+x^2}$;

   (C) $(\ln(1+x^2))^x\left\{\ln\left(\ln(1+x^2)\right)+\frac{2x^2}{(1+x^2)\ln(1+x^2)}\right\}$;

   (D) $(\ln(1+x^2))^x\left\{\ln\left(\ln(1+x^2)\right)+\frac{2x}{(1+x^2)\ln(1+x^2)}\right\}$.

3. Consider a function

$$f(x)=\begin{cases}\ln x - x, & x\ge 1,\\ \alpha x+\beta, & x<1,\end{cases}$$

   where α and β are constants. If $f(x)$ is **differentiable** on all real numbers, then $(\alpha,\beta)=$

   (A) $(1,0)$; (B) $(-1,0)$; (C) $(0,-1)$; (D) $(0,1)$.

4. $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(1+\frac{j}{n}\right)^2 =$

(A) $\int_0^1 x^2\,dx$; (B) $\int_1^2 (1+x^2)\,dx$;

(C) $\int_0^1 x\sqrt{x^2+1}\,dx$; (D) $\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$.

5. The **length** of the polar curve $r = 1+\sin\theta$ for $0 \le \theta \le \frac{\pi}{2}$ is

(A) 8; (B) $2\sqrt{2}$; (C) 4; (D) $4\sqrt{2}$.

6. The **volume** of the solid obtained by rotating the region bounded by the given curves $y = \sec x$ and $y = \sqrt{2}$ for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ about the x-axis is

(A) $\frac{\pi^2}{2}$; (B) $4\pi - \pi^2$; (C) $\pi(\pi-1)$; (D) $\pi(\pi-2)$.

7. Assume that n is a positive integer. The value of the definite integral $\int_0^1 (\ln x)^n dx$ is

(A) $(-1)^n(n!)$; (B) $(-1)^n n^n$; (C) $-(n!)$; (D) $(-1)^n(n-1)!$.

8. Let g be a differentiable function defined in $(-\infty,\infty)$ and $g'(x) > 0$ for all real numbers x. Suppose that $F(x)$ is the average value of g on the interval $[0,x]$ where $x > 0$. How many **critical** numbers on $(0,\infty)$ does F have ?

(A) 0; (B) 1; (C) 2; (D) 3.

9. The exact value of the improper integral

$$\int_0^{+\infty} \frac{\ln x}{1+x^2} dx \quad \text{is}$$

(A) $+\infty$; (B) $\frac{1}{2}$; (C) 0; (D) e.

10. Assume that

$$F(s) = \int_0^1 |\ln|s-t||dt.$$

The absolute **maximum** value of $F(s)$ on the interval $[0,1]$ is

(A) $\frac{1}{2} - \ln 3$; (B) $1+\ln 2$; (C) $1+\frac{3}{2}\ln 2$; (D) $\frac{4}{5} + \ln(\frac{3}{2})$.

⊙ **Part2：多選擇題**

(Multiple-Choice Questions with More Than One Correct Answers)

(多選五題，每題五分，共二十五分。答錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)

(5 questions, each question is worth 5 points for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)

Example:

Answer: ABC

Student: ABC => 5 points , AB => 3 points, AD => 0 points, ABD => 0 points

11. Let $f(x) = -|\sin|x - \frac{\pi}{6}||$. Which of the following statements are **TRUE**?

(A) $f'(x)$ exists at $x = 0$;

(B) f has a local maximum at $x = -\frac{4}{3}\pi$;

(C) $f''(x)$ does not exist at $x = (n + \frac{1}{6})\pi$ for all integers n;

(D) f has an inflection point.

12. Consider the following function:

$$f(x) = \begin{cases} x^n \cos(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where n is a positive integer. Which of the following statements are **TRUE**?

(A) For any positive integer n, f is continuous on all real numbers;

(B) If $n = 1$, f is differentiable on all real numbers;

(C) If $n \geq 3$, f' is continuous at $x = 0$;

(D) For any positive integer n, $y = f(x)$ has a slant asymptote.

13. Consider three functions $f_1(x) = e^{2x}(2 - \sin(e^x)) - 1$, $f_2(x) = 2x$ and $f_3(x) = 2\sin x$. Let A_i be the area bounded by x-axis, $x = 0$, $x = 1$ and $y = f_i(x)$ where $i = 1, 2, 3$. Which of the following statements are **TRUE**?

(A) $A_2 > A_1$; (B) $A_3 > A_2$; (C) $A_1 > A_3$; (D) $A_1 + A_3 > 2A_2$.

14. Which of the following statements are **TRUE**?

(A) Let f be a continuous function on the interval $[-1, 1]$, then

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(-x)\,dx;$$

(B) Let $f(x) \geq 0$ be a continuous function on the interval $[-1, 1]$ satisfying $\int_{-1}^{1} f(x)\,dx = 0$. Then $f = 0$ on $[-1, 1]$;

(C) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan x}{1 + x^4}\,dx = \frac{\pi}{2};$

(D) $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$ converges.

15. Which of the following statements are **TRUE**?

(A) Let f be defined on the interval (a, b) and $c \in (a, b)$ be a number. If f' is continuous at $x = c$, then f is differentiable at $x = c$;

(B) Let f be bounded and differentiable on the interval $(1, \infty)$. If $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f'(x) = 0$;

(C) Let f be defined on the interval (a, b) and $c \in (a, b)$ be a number. If $\lim_{h\to 0} |f(c + h) - f(c - h)| = 0$, then

$$\lim_{h\to 0} |f(c + h) - f(c)| = 0;$$

(D) Let f be defined on the interval $[a, \infty)$. If $f''(x) \geq 0$ for all $x \in (a, \infty)$ and $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x} = 1$, then $\lim_{x\to\infty} f'(x) = 1$.

◎Part3：計算/證明題 (Questions of calculations and proofs)
(Answer the problems as thoroughly as possible. Be sure to include all your work. Partial credits will be given even if the answer is not fully correct.)

1. In this problem, we would like to study the following improper integral

$$I_n = \int_1^\infty \frac{dx}{x(x+1)(x+2)\cdots(x+n)}$$

where n is a positive integer Please answer the following questions:

(A) (4 points) Evaluate the following indefinite integral

$$\int \frac{dx}{x(x+1)(x+2)}.$$

(B) (4 points) Evaluate the following improper integral.

$$\int_1^\infty \frac{dx}{x(x+1)(x+2)}.$$

(C) (2 points) Please guess and write down an explicit expression for I_n where n is a positive integer in terms of the combintorical numbers $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ and $n! = 1, 2, 3, \cdots n$(without the proof).

2. In this problem, we will study how to sketch the curve $y = f(x) = \dfrac{2x^3}{x^2-4}$. Please answer the following questions:

(A) (3 points) Find all horizontal, vertical and slant asymptotes, if any, of the curve $y = f(x)$.

(B) (5 points) Find the intervals of increase and decrease respectively and all local maximum and minimum values of the curve $y = f(x)$.

(C) (4 points) Find the intervals of concavity and the inflection points of the curve $y = f(x)$.

(D) (3 points) Sketch the curve $y = f(x)$ by indicating asymptotes, local maximum/minimum and the inflection points on the figure.