《一百一十學年度第一學期微積分會考答案卷》(A 卷)

姓名					老師			
學號							系別		系

總分(第一部份~第三部份)	初閱		複閱	

第一、二、三部份合計總分	

第一部份：單選擇題

1	**A**	**2**	**A**	**3**	**B**	**4**	**C**	**5**	**D**
6	**C**	**7**	送分	**8**	**B**	**9**	**C**	**10**	**B**

初閱		評分
複閱		

第二部份：複選擇題

11	**AD**	**12**	**BC**	**13**	**BD**	**14**	**AC**	**15**	**AD**

初閱		評分
複閱		

Q1

(a) $\sin(x) \le 1$ and $t > 0$, so $\sin^t(x) \le 1$. (**1pt**)
For $x \in [0, \pi/2]$, $0 \le \cos(x) \le 1$, so $\sin^t(x)\cos(x) \le \sin^t(x)$. (**1pt**)

Remark 0.1. **1 pt** for each inequality.

(b)

$$= \left.\frac{\sin^{t+1}(x)}{t+1}\right|_0^{\frac{\pi}{2}} \quad (\textbf{2pt})$$

$$= \frac{1}{t+1} \quad (\textbf{1pt})$$

(c)

$$\left(\int_0^{\pi/2} \sin^t(x)\cos(x)\,dx\right)^{1/t} \le \left(\int_0^{\pi/2} \sin^t(x)\,dx\right)^{1/t} \le \left(\int_0^{\pi/2} 1\,dx\right)^{1/t} \quad (\textbf{1pts})$$

$$\lim_{t\to\infty} \frac{1}{t}\ln\left(\frac{1}{t+1}\right) = 0, \quad \to \quad \lim_{t\to\infty}\left(\int_0^{\pi/2} \sin^t(x)\cos(x)\,dx\right)^{1/t} = \lim_{t\to\infty}\left(\frac{1}{t+1}\right)^{1/t} = 1. \quad (\textbf{2pts})$$

Remark 0.2. Calculation is required.

$$\lim_{t\to\infty}\left(\int_0^{\pi/2} 1\,dx\right)^{1/t} = \lim_{t\to\infty}\left(\frac{\pi}{2}\right)^{1/t} = 1. \quad (\textbf{1pts})$$

By the Squeez Theorem, we have

$$\lim_{t\to\infty}\left(\int_0^{\pi/2} \sin^t(x)\,dx\right)^{1/t} = 1. \quad (\textbf{1pts})$$

Q2

(a) If f is continuous on $[a,b]$, there there is $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(**2pt**)

Remark 0.3. If written as $c \in (a,b)$ is ok, get **2pt**.

(b) $F''(x) = \int_1^x f(u)\,du + xf(x)$.
$F''(\frac{1}{2}) = \int_1^{1/2} f(u)\,du + \frac{1}{2}f(\frac{1}{2})$. (**2pt**)
According to MVT, there exits $\xi \in [\frac{1}{2}, 1]$ such that $\int_1^{1/2} f(u)\,du = -\frac{1}{2}f(\xi)$. (**2pt**)
Since f is an increasing function, $F''(\frac{1}{2}) = \frac{1}{2}(-f(\xi) + f(\frac{1}{2})) \leq 0$. (**2pt**)

(b') $F''(x) = \int_1^x f(u)\,du + xf(x)$.
$F''(\frac{1}{2}) = \int_1^{1/2} f(u)\,du + \frac{1}{2}f(\frac{1}{2})$. (**2pt**)
We have $F''(\frac{1}{2}) = \int_1^{1/2} \left(f(u) - f(\frac{1}{2})\right)\,du$. (**2pt**)
Since f is an increasing function, $f(u) - f(\frac{1}{2}) > 0$, so $F''(\frac{1}{2}) \leq 0$. (**2pt**)

(b") $F''(x) = \int_1^x f(u)\,du + xf(x)$.
$F''(\frac{1}{2}) = \int_1^{1/2} f(u)\,du + \frac{1}{2}f(\frac{1}{2})$. (**2pt**)
Since f is an increasing function, $\int_{1/2}^1 f(u)\,du \leq \frac{1}{2}f(\frac{1}{2})$, (**2pt**)
so $F''(\frac{1}{2}) = \int_1^{1/2} f(u)\,du + \frac{1}{2}f(\frac{1}{2}) \leq 0$. (**2pt**)

(c) $F'''(x) = 2f(x) + xf'(x)$. (**1pt**)
Since f is an increasing function, $f' > 0$. (**1pt**)
$x \in (0,1)$, $f > 0$ and $f' > 0$,
so $F''' > 0$, therefore F'' is an increasing function. (**1pt**)

(c') Assume $x > y$, we have $F''(x) - F''(y) = \int_y^x f(u)\,du + (xf(x) - yf(y))$.
Since $f > 0$, $\int_y^x f(u)dx > 0$. (**1pt**)
Since f is an increasing function, $xf(x) - yf(y) > 0$. (**1pt**)
Therefore F'' is an increasing function. (**1pt**)

(c") $f(x)$ is an increasing function, so $\int_1^x f(u)\,du$ is an increasing function. (**1pt**)
x and $f(x)$ are increasing functions, so $xf(x)$ is an increasing function. (**1pt**)
Therefore, $F''(x) = \int_1^x f(u)\,du + xf(x)$ is the sum of two increasing functions, which is also an increasing function. (**1pt**)

(d) Suppose $F''(\frac{1}{2}) = 0$, done. (**1pt**)

(Existence) $F''(1) = \int_1^1 f(u)\,du + f(1) = f(1) > 0$;
Also we have $F''(\frac{1}{2}) < 0$, by Intermediate Value Theorem, there exits $\xi' \in (\frac{1}{2}, 1)$ such that $F''(\xi') = 0$. (**2pt**)

(Uniqueness) Furthermore, F'' is increasing (by (c)), so there exists exactly one root of F. (**1pt**)