一百一十學年度第一學期微積分會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 7 頁。

(3) 試卷包括選擇題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將做為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part1：單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. If $\lim_{x\to 2} \dfrac{x-2}{\sqrt{ax+b}-3} = 9$, then $a-b=$

   (A) -7; (B) 0; (C) 5; (D) 7.

2. We consider the region $\Omega = \{(x,y)\colon x \geq 1,\ \frac{1}{x^2} \leq y \leq \frac{1}{x}\}$ and rotate it about the x-axis. The volume of the resulting solid equals to

   (A) $\frac{2\pi}{3}$; (B) $\frac{3\pi}{4}$; (C) $\frac{4\pi}{5}$; (D) ∞.

3. Find the area of the region that is inside the curve $r = 2 + 2\cos\theta$ and outside the curve $r = 3$.

   (A) $\pi - 1$; (B) $\dfrac{9\sqrt{3}}{2} - \pi$; (C) $2(\pi - 2)$; (D) $\dfrac{\pi+1}{2}$.

4. Find the value of a such that curves $y = 2^x$ and $y = ax^2$ are tangent at some points.

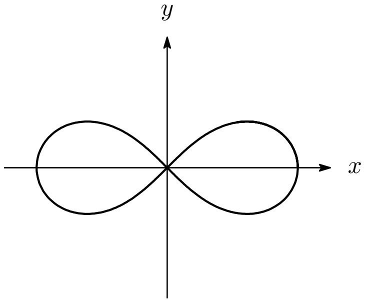
   (A) $\frac{e^2}{4}$ (B) $e^2 \ln 2$; (C) $e^2(\ln\sqrt{2})^2$; (D) e^2.

5. Which of the followings is the graph of the parametric equations

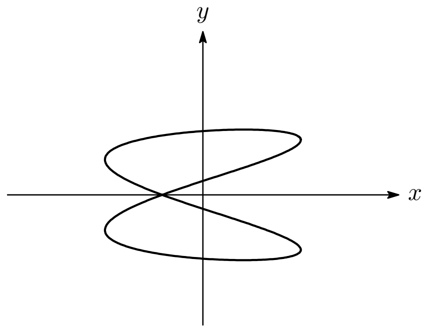
$$x(t) = \frac{3at}{1+t^3}, \quad y(t) = \frac{3at^2}{1+t^3},$$

where $a > 0$ and $t \neq -1$?

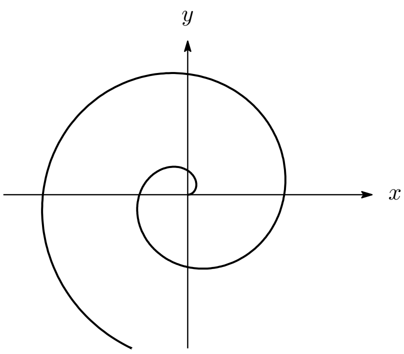
(A)



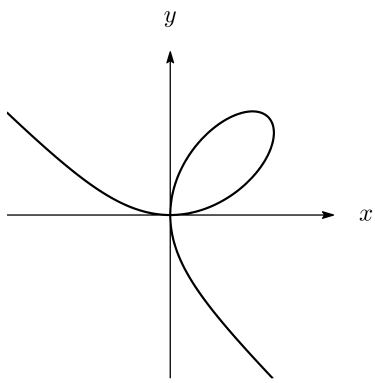
(B)



(C)



(D)



6. Let $f(x) = \left(1 + \frac{b}{x}\right)^x$, where $b > 0$ and $x > 0$. $f'(b) =$

(A) $\frac{2^b}{2}$; (B) $-\frac{2^b}{2}$; (C) $2^b\left(\ln 2 - \frac{1}{2}\right)$; (D) $2^b\left(\ln 2 + \frac{1}{2}\right)$.

7. Let $f(x)$ be an odd function defined on $\mathbb{R}$. If $f'(x)$ is continuous and $f(1) = 1$, then $\int_0^2 \cos(\pi x)\, f'(1-x)\, dx =$

(A) -2; (B) 0; (C) 1; (D) 2.

8. Suppose that f is continuous on $\mathbb{R}$ and satisfies the identity

$$\int_0^x f(t)dt + \int_0^{x^3} \sin(\sqrt[3]{t})f(\sqrt[3]{t})dt = (x-\pi)^3 + 2\sin x + \pi^3, \quad \text{for any } x \in \mathbb{R}.$$

Then $f(\pi) =$

(A) 0; (B) -2; (C) $\frac{\pi}{2}$; (D) 16.

9. Compute the limit: $\lim_{n\to\infty} \frac{\sqrt[n]{n\times(n-1)\times\cdots\times 1}}{n}$, where $n \in \mathbb{N}$.

(A) 0; (B) e^{-2}; (C) e^{-1}; (D) 1.

10. Given a function $f(x) = \pi \sin x$. For any real number $x \neq 0$, three points $P(x, 0)$, $O(0, 0)$ and $Q(x, f(x))$ can determine the angle $\theta = \angle POQ$. Find the rate of change of θ with respect to x at $x = 3\pi$.

(A) $-\frac{1}{9}$; (B) $-\frac{1}{3}$; (C) 0; (D) $\frac{1}{9}$.

◎ **Part2：多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。答錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)

(5 questions, each question is worth 5 points for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)

Example:
Answer: ABC
Student: ABC => 5 points , AB => 3 points, AD => 0 points, ABD => 0 points

11. Suppose that $f(x)$ is an even function. Which of the following statements are always **TRUE**?

(A) If $f'(0)$ exists, then $f'(0) = 0$.

(B) $f(x)$ has a local maximum or minimum value at $x = 0$.

(C) $f(x)$ can not be an odd function.

(D) $(f(x))^3$ is also an even function.

12. Which of the following statements are always **TRUE**?

(A) Let $f(x)$ be a function defined on $\mathbb{R}$. If $f(1) > 0$ and $f(4) < 0$, then there is a number c between 1 and 4 such that $f(c) = 0$.

(B) If $f(x)$ is continuous on $\mathbb{R}$, so is $|f(x)|$.

(C) If $f'(c)$ exists, then $f(x)$ is continuous at $x = c$.

(D) If $\lim_{x\to a} |f(x)| = \infty$, then $x = a$ is a vertical asymptote of $f(x)$.

13. Let

$$f(x) = \begin{cases} (x+1)\, 2^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & \text{if } x \in [-1,0) \cup (0,1], \\ 0 & \text{if } x = 0\,. \end{cases}$$

Which of the following statements are **TRUE**?

(A) $f(x)$ is continuous from the left at $x = 0$.

(B) $f(x)$ is differentiable except at $x = 0$.

(C) $f(x)$ attains its absolute maximum value.

(D) $f(x)$ attains its absolute minimum value.

14. Which of the following statements are **TRUE**?

(A) $\displaystyle\int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx$ is convergent.

(B) $\displaystyle\int_{-1}^{1} \frac{\tan^{-1} x}{x^2} dx$ is convergent.

(C) $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^{101}} dx = \frac{1}{100}$.

(D) $\displaystyle\int_1^{e} (\ln x)^2 dx = e + 2$.

15. Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^4}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Which of the following statements are **TRUE**?

(A) $f(x)$ is differentiable at $x = 0$.

(B) $f'(x)$ is a continuous function.

(C) $y = f(x)$ has infinitely many intersections with the curve $y = x$.

(D) $y = 0$ is a horizontal asymptote of $y = f(x)$.

◎Part3：計算/證明題 (Questions of calculations and proofs)
(答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分。如果計算錯誤，則酌給部分分數。如果只有答案對，但觀念錯誤，或是過程不合理，則無法得到分數。)
(Answer the problems as thoroughly as possible. Be sure to include all your work. Partial credits will be given even if the answer is not fully correct.)

1. **(1). (2 points)** Prove that the following inequality holds for any $t > 0$:

$$\sin^t(x)\cos(x) \le \sin^t(x) \le 1, \quad x \in [0, \frac{\pi}{2}].$$

(2). (3 points) Evaluate

$$\int_0^{\frac{\pi}{2}} \sin^t(x)\cos(x)\,dx.$$

(3). (5 points) Use **the Squeeze Theorem** to determine the value of the following limit:

$$\lim_{t\to\infty}\left(\int_0^{\frac{\pi}{2}} \sin^t(x)\,dx\right)^{\frac{1}{t}}.$$

2. Suppose that $f(x)$ is a differentiable function on $\mathbb{R}$. We further assume that $f(x)$ is positive ($f(x) > 0$) and increasing. Let

$$F(x) = \int_0^x g(t)\,dt, \quad g(t) = t\int_1^t f(u)\,du,$$

and it is known that g is a continuous function.

(1). (2 points) State the **Mean Value Theorem for integrals**.

(2). (6 points) Show that $F''(\frac{1}{2}) \le 0$.

(3). (3 points) Show that $F''(x)$ is an increasing function on $(0, 1)$.

(4). (4 points) Show that $F''(x)$ has exactly one root on $[\frac{1}{2}, 1]$.