《一百一十一學年度第一學期微積分會考答案卷》(A 卷)

姓名					老師			
學號					系別			系

總分(第一部份~第三部份)	初閱		複閱	

第一、二、三部份合計總分	

第一部份：單選擇題

1	**B**	**2**	**D**	**3**	**A**	**4**	送分	**5**	**C**
6	**D**	**7**	**A**	**8**	**C**	**9**	**B**	**10**	**C**

初閱		評分
複閱		

第二部份：複選擇題

11	**AC**	**12**	**ACD**	**13**	**CD**	**14**	**BD**	**15**	**ACD**

初閱		評分
複閱		

1. Let $G(x) = \displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$.

(A) (3 points) Prove that, for any $a > 0$, the improper integral $\displaystyle\int_0^\infty e^{-at}dt$ is convergent.

[Solution]

$$\begin{aligned}\int_0^\infty e^{-at}dt &= \lim_{n\to\infty}\int_0^n e^{-at}dt = \lim_{n\to\infty}\left(\frac{e^{-at}}{-a}\Bigg|_{t=0}^{t=n}\right) \qquad (1\text{pt})\\ &= \lim_{n\to\infty}\frac{e^{-an}-1}{-a} \qquad (1\text{pt})\\ &= \frac{1}{a}, \qquad (1\text{pt})\end{aligned}$$

so the improper integral is convergent.

(B) (3 points) Prove that for any $n \in \mathbb{N}$,

$$\lim_{t\to\infty} t^{n-1}e^{-t/2} = 0.$$

[Solution]

As $n = 1$, $\lim\limits_{t\to\infty} t^{n-1}e^{-t/2} = \lim\limits_{t\to\infty} e^{-t/2} = 0.$ (1pt)

As $n \geq 2$,

$$\begin{aligned}\lim_{t\to\infty} t^{n-1}e^{-t/2} &= \lim_{t\to\infty}\frac{t^{n-1}}{e^{t/2}} \qquad \text{[by using the L'Hospital rule]}\\ &= \lim_{t\to\infty}\frac{(n-1)t^{n-2}}{\frac{1}{2}e^{t/2}} \qquad (1\text{pt})\\ &\vdots \qquad \text{[by using the L'Hospital rule again]}\\ &= \lim_{t\to\infty}\frac{(n-1)(n-2)\cdots 1\cdot t^0}{(\frac{1}{2})^{n-1}e^{t/2}} \qquad (1\text{pt})\\ &= 0.\end{aligned}$$

(C) (4 points) Prove that $G(x)$ is well-defined on $[1, \infty)$, i.e., to prove the improper integral $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$ is convergent for any $x \geq 1$. [Hint: use (A), (B), and the Comparison Test for Improper Integrals].

[Solution]

By (B), there exists $M > 0$ such that

$$0 \leq t^{n-1}e^{-t} \leq e^{t/2}e^{-t} = e^{-t/2}, \quad t \geq M. \qquad (1\text{pt})$$

On the other hand, thanks to (A) with $a = 1/2$, $\int_M^\infty e^{-t/2}\,dt$ converges. (1pt)

Then, by the Comparison Test for Improper Integrals, we see that $\int_M^\infty t^{n-1}e^{-t}dt$ converges for all $n \in \mathbb{N}$, and so does $\int_0^\infty t^{n-1}e^{-t}dt$ for all $n \in \mathbb{N}$. (1pt)

For each $x \geq 1$, we have

$$0 \leq t^{x-1}e^{-t} \leq t^{[x]}e^{-t}, \qquad (1\text{pt})$$

where $[x]$ is the greatest integer function. Since $\int_0^\infty t^{[x]}e^{-t}dt$ converges, we see that $\int_0^\infty t^{x-1}e^{-t}dt$ also converges by the Comparison Test for Improper Integrals.

2. Let $h(x) = \dfrac{1+x^2}{1+x^4}$ for $x > 0$.

(A) (10 points) Evaluate $\displaystyle\int h(x)\,dx$. [Hint: use the substitution $t = x - x^{-1}$ for $x > 0$]

[Solution]

Let $t = x - x^{-1}$ for $x > 0$, then $dt = (1 + x^{-2})\,dx = \dfrac{x^2+1}{x^2}\,dx$. (2pt)

And $t^2 = (x - x^{-1})^2 = x^2 + x^{-2} - 2$, i.e., $t^2 + 2 = x^2 + x^{-2} = \dfrac{x^4+1}{x^2}$. (2pt)

$$
\begin{aligned}
\int \frac{1+x^2}{1+x^4}\,dx &= \int \frac{x^2}{1+x^4}\,\frac{1+x^2}{x^2}\,dx \qquad \text{(1pt)}\\
&= \int \frac{1}{t^2+2}\,dt \qquad \text{(1pt)}\\
&= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C \qquad \text{(2pt)}\\
&= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}(x - x^{-1})\right) + C \qquad \text{(2pt)}
\end{aligned}
$$

(B) (5 points) Evaluate $\displaystyle\int_1^\infty h(x)\,dx$.

[Solution]

$$
\begin{aligned}
\int_1^\infty h(x)\,dx &= \lim_{L\to\infty}\int_1^L h(x)\,dx \qquad \text{(1pt)}\\
&= \lim_{L\to\infty}\left\{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}(x - x^{-1})\right)\Bigg|_{x=1}^{x=L}\right\} \qquad \text{(1pt)}\\
&= \lim_{L\to\infty}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}(L - L^{-1})\right) \qquad \text{(1pt)}\\
&= \frac{1}{\sqrt{2}}\,\frac{\pi}{2} \qquad \text{(2pt)}
\end{aligned}
$$