一百一十一學年度微積分甲 (一) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 7 頁 (另再加 2 頁空白頁)。

(3) 試卷包括選擇題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Let $f(x) = x - \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. How many roots does f have?

   (A) 0. (B) 1. (C) 2. (D) 3.

2. Consider the following function

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   Find the value of $f'(0)$.

   (A) Does not exist. (B) 0. (C) 1. (D) $\dfrac{1}{2}$.

3. Let f be a continuous function with $\lim_{x\to 12} f(x) = 0$ and $\lim_{x\to 12} f'(x) = 1000$. Find the value of

$$\lim_{x\to 12} \frac{\int_{12}^{x} (t \int_t^{12} f(\theta)\, d\theta)\, dt}{(12-x)^3}.$$

   (A) 2000. (B) 1000. (C) 0. (D) Does not exist.

4. Suppose that a function $f : \mathbb{R} \to [0,\infty)$ is continuous and a function g is defined by

$$g(x) = \begin{cases} \dfrac{\int_0^x t f(t)\, dt}{\int_0^x f(t)\, dt}, & \text{if} \quad x \neq 0, \\ 0\,, & \text{if} \quad x = 0. \end{cases}$$

   Find the value of $g'(0)$.

   (A) 0. (B) $\frac{1}{2}$. (C) 1. (D) Does not exist.

5. Suppose that $\lim_{n\to\infty}\left\{\sin(\frac{\pi}{n})\sum_{k=1}^{n}\left(\cos(\frac{k\pi}{4n})\right)^{-1}\right\} = \alpha\cdot\ln(\sqrt{2}+1)$. Find the value of α.

(A) 1. (B) 2. (C) 4. (D) 8.

6. Suppose that $f(x) = \int_0^x e^{-t^2+2t}\,dt$. Find the integral $\int_0^1 (x-1)^2 f(x)\,dx$.

(A) $-\frac{1}{3}(e-2)$. (B) $\frac{1}{3}(e-2)$. (C) $-\frac{1}{6}(e-2)$. (D) $\frac{1}{6}(e-2)$.

7. Find the value of $\int_2^\infty \frac{1}{x\ln(1+x^3)}\,dx$.

(A) ∞. (B) e. (C) 1. (D) $\frac{1}{2}$.

8. Let $V(t)$ be the volume of the solid obtained by rotating the region

$$A(t) := \{(x,y) : 0 < x \le t, 0 \le y \le \frac{\sin(x)}{x}\}$$

about the y-axis. If the limit $\lim_{t\to 0^+}\frac{V(t)}{t^a} = L$ exists, then (a, L) is?

(A) $(1,\pi)$. (B) $(1,\pi/2)$. (C) $(2,\pi)$. (D) $(2,\pi/2)$.

9. Let R be the region inside the ellipse $\frac{x^2}{4}+y^2 = 1$ and outside the circle $x^2+y^2 = 1$ and let S be the solid obtained by rotating R about the x-axis. Find the volume of S.

(A) $\frac{2\pi}{3}$. (B) $\frac{4\pi}{3}$. (C) $\frac{8\pi}{3}$. (D) $\frac{16\pi}{3}$.

10. Find the length of the polar curve $r = e^{-\theta/2}$, $0 \le \theta < \infty$.

(A) 1. (B) $\sqrt{2}$. (C) $\sqrt{5}$. (D) ∞.

⊙ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points, for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Suppose that the function f is defined by $f(x) = \lim_{n\to\infty} \dfrac{x^{n+2} - x^{-n}}{x^n + x^{-n}}$ for $x > 0$. Which of the following statements are **TRUE**?

    (A) $\lim_{x\to 0^+} f(x)$ exists.
    (B) f is continuous at 1.
    (C) $\lim_{x\to 1^+} f(x) = 1$.
    (D) $f(x) = x^2$ if $0 < x < 1$.

12. Let a, b be constants and f be a function which is continuous at 0 and satisfies

$$\lim_{x\to 0} \frac{e^{af(bx)} - 1}{x} = 1.$$

    Which of the following statements are **TRUE**?

    (A) $ab \neq 0$. (B) $a + b = 1$. (C) $f(0) = 0$. (D) f is differentiable at 0.

13. Let f be a function defined in $(-1, 1)$. Which of the following statements are **TRUE**?

    (A) If $f'(x) = 0$ for all $x \in (-1, 1)\setminus\{0\}$ and $f(0) = 3$, then $f(x) = 3$ for all $x \in (-1, 1)$.
    (B) If $\lim_{h\to 0} |f(h) - f(-h)| = 0$, then $\lim_{h\to 0} |f(h) - f(0)| = 0$.
    (C) If f' is differentiable on $(-1, 1)$, then f' is continuous on $(-1, 1)$.
    (D) If f''' is continuous on $(-1, 1)$, $f'(0) = f''(0) = 0$ and $f'''(0) > 0$, then f has an inflection point at $x = 0$.

14. Which of the following statements are **TRUE**?

(A) If $f(x)$ is not continuous on $[a, b]$, then $f(x)$ is not integrable on $[a, b]$.
(B) If $f(x)$ is differentiable on (a, b), and has an absolutely maximum at $c \in (a, b)$, then $f'(c) = 0$.
(C) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with the slant asymptotes $y = x$ and $y = -x$. Then, it follows that $\lim_{x\to\infty} f'(x)$ exists.
(D) If $f(x)$ is continuous on $\mathbb{R}$, differentiable on $(-\infty, 0)$ and $(0, \infty)$, and $\lim_{x\to 0} f'(x)$ exists, then $f'(x)$ is continuous at $x = 0$.

15. Consider the curve $y = \sin x$. For $x > 0$, let $C(x)$ be the curve from $(0, 0)$ to $(x, \sin x)$. Let $s(x)$ be the length of $C(x)$ and $A(x)$ be the area of the surface obtained by rotating $C(x)$ about the x-axis. Which of the following statements are **TRUE**?

(A) $s(x) = \displaystyle\int_0^x \sqrt{1 + \cos^2 t}\, dt$.
(B) $A(x) = \displaystyle\int_0^x 2\pi \sin t \sqrt{1 + \cos^2 t}\, dt$.
(C) $\lim_{x\to\infty} s(x) = \infty$.
(D) $\displaystyle\lim_{x\to\infty} \frac{s(x)}{x}$ exists.

⊙ **Part 3: 計算/證明題 (Questions of calculations and proofs)**
(兩個題組，共二十五分。)
(2 questions are worth for 25 points in total.)
(答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分。如果
(計算錯誤，則酌給部分分數。如果只有答案對，但觀念錯誤，或是過程不合理，
(則無法得到分數。)
(Answer the problems as thoroughly as possible. Be sure to include all your work. Partial credit will be given even if the answer is not fully correct.)

1. Let $G(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$.
   (A) (3 points) Prove that, for any $a > 0$, the improper integral $\int_0^\infty e^{-at}dt$ is convergent.
   (B) (3 points) Prove that for any $n \in \mathbb{N}$,
   $$\lim_{t\to\infty} t^{n-1}e^{-t/2} = 0.$$
   (C) (4 points) Prove that $G(x)$ is well-defined on $[1, \infty)$, i.e., to prove the improper integral $\int_0^\infty t^{x-1}e^{-t}\,dt$ is convergent for any $x \geq 1$. [Hint: use (A), (B), and the Comparison Test for Improper Integrals].

2. Let $h(x) = \dfrac{1+x^2}{1+x^4}$ for $x > 0$.
   (A) (10 points) Evaluate $\int h(x)\,dx$. [Hint: use the substitution $t = x - x^{-1}$ for $x > 0$]
   (B) (5 points) Evaluate $\int_1^\infty h(x)\,dx$.