《一百一十一學年度第二學期微積分會考答案卷》(A 卷)

姓名		老師	
學號		系別	系

總分(第一部份~第三部份)	初閱		複閱	

第一、二、三部份合計總分	

第一部份：單選擇題

1	**C**	**2**	**B**	**3**	**B**	**4**	**D**	**5**	**A**
6	**C**	**7**	**D**	**8**	**A**	**9**	**C**	**10**	**D**

初閱		評分
複閱		

第二部份：複選擇題

11	**ABD**	**12**	**ABC**	**13**	**AB**	**14**	**(送分)**	**15**	**AC**

初閱		評分
複閱		

一百一十一學年度微積分甲 (二) 會考試題
計算/證明題參考解答

1. Let Ω be the xy-plane; it means $\Omega = \{(x,y) : -\infty < x, y < \infty\}$. Given a double integral

$$\iint_{\Omega} e^{-(x^2+2xy+5y^2)}\, dA.$$

(A) (4 points) Let $u = x + y$ and $v = 2y$. Evaluate the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

[Solution 1]

$$\begin{cases} u = x+y \\ v = 2y \end{cases} \Rightarrow \begin{cases} x = u - v/2 \\ y = v/2 \end{cases} \qquad \text{(1pt)}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1/2 \\ 0 & 1/2 \end{vmatrix} \qquad \text{(1pt)}$$

$$= \tfrac{1}{2} \qquad \text{(2pt)}$$

[Solution 2]

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \qquad \text{(1pt)}$$

$$= 2 \qquad \text{(1pt)}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \tfrac{1}{2} \qquad \text{(2pt)}$$

(B) (4 points) Use the change of variables from (A) to write down the integral form with variables u and v for

$$\iint_{\Omega} e^{-(x^2+2xy+5y^2)}\, dA.$$

[Solution]

Let $T(u,v) = (x,y) = \left(u - \dfrac{v}{2}, \dfrac{v}{2}\right)$. Then, $T : S \to \Omega$, where $S = \mathbb{R}^2 = \Omega$. It's clear that T is a C^1 transformation. (1pt)

$$\text{Therefore, } \iint_{\Omega} e^{-(x^2+2xy+5y^2)}\, dA = \iint_{\Omega} e^{-(x+y)^2-(2y)^2}\, dA \qquad \text{(1pt)}$$

$$= \iint_{S} e^{-(u^2+v^2)}\, \frac{1}{2}\, dA \qquad \text{(2pt)}$$

$$\left[\text{or } \tfrac{1}{2}\iint_{\Omega} e^{-(u^2+v^2)}\, dA \ \text{ or } \ \tfrac{1}{2}\iint_{\mathbb{R}^2} e^{-(u^2+v^2)}\, dA \ \text{ or } \ \tfrac{1}{2}\iint_{\mathbb{R}^2} e^{-(u^2+v^2)}\, dudv \ \text{ or } \ \tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}\, dudv \right]$$

(C) (7 points) We have known the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Use (A) and (B) to evaluate the value of the double integral

$$\iint\limits_{\Omega} e^{-(x^2+2xy+5y^2)}\,dA.$$

[Solution]

From #50 in §15.3, p.1069, we know that

$$\begin{aligned}\iint\limits_{\mathbb{R}^2} e^{-(u^2+v^2)}\,dA &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}\,dudv \qquad \text{(3pt)}\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-u^2}\,du\right) e^{-v^2}\,dv \qquad \text{(1pt)}\\ &= \int_{-\infty}^{\infty} \sqrt{\pi}\,e^{-v^2}\,dv = \sqrt{\pi}\cdot\sqrt{\pi} = \pi. \qquad \text{(1pt)}\end{aligned}$$

Therefore, $\displaystyle\iint\limits_{\Omega} e^{-(x^2+2xy+5y^2)}\,dA = \iint\limits_{S} e^{-(u^2+v^2)}\,\frac{1}{2}\,dA = \frac{\pi}{2}.$ (2pt)

2. Let $\{a_n\}$ be a sequence of positive numbers and σ be a positive number.

(A) (3 points) If there exists some positive integer N such that

$$(\ln n)^{-1} \cdot \ln\left(\frac{1}{a_n}\right) \geq 1+\sigma \text{ holds for } n \geq N.$$

Prove that $\sum_{n=1}^{\infty} a_n$ is convergent.

[Solution]

It's equivalent to $\ln\left(\frac{1}{a_n}\right) \geq (1+\sigma)\ln n = \ln\left(n^{1+\sigma}\right)$ for all $n \geq N$. Then, $\frac{1}{a_n} \geq n^{1+\sigma}$ for all $n \geq N$. i.e., $0 < a_n \leq \frac{1}{n^{1+\sigma}}$ for all $n \geq N$. (1pt)

Since $\sum_{n=N}^{\infty} \frac{1}{n^{1+\sigma}}$ is convergent (conv. as p-series with $p > 1$), (1pt)

by the comparison test, $\sum_{n=1}^{\infty} a_n$ is convergent. (1pt)

(B) (3 points) If there exists some positive integer N such that

$$(\ln n)^{-1} \cdot \ln\left(\frac{1}{a_n}\right) \leq 1 \text{ holds for } n \geq N.$$

Prove that $\sum_{n=1}^{\infty} a_n$ is divergent.

[Solution]

It's equivalent to $\ln\left(\frac{1}{a_n}\right) \leq \ln n$ for all $n \geq N$. Then, $\frac{1}{a_n} \leq n$ for all $n \geq N$.

i.e., $a_n \geq \frac{1}{n}$ for all $n \geq N$. (1pt)

Since $\sum_{n=N}^{\infty} \frac{1}{n}$ is divergent (div. as p-series with $p \leq 1$), (1pt)

by the comparison test, $\sum_{n=1}^{\infty} a_n$ is divergent. (1pt)

(C) (4 points) By using the results in (A), (B) or otherwise, find all $b \in (0, \infty)$ such that $\sum_{n=1}^{\infty} b^{\ln(n^3+1)}$ is convergent.

[Solution]

Take $a_n = b^{3\ln n} = b^{\ln n^3}$ for $n \geq 1$.

Then, $(\ln n)^{-1} \cdot \ln\left(\frac{1}{a_n}\right) = (\ln n)^{-1} \cdot \ln\left(b^{-3\ln n}\right) = \ln\left(b^{-3}\right)$. (1pt)

From the results of (A) and (B),

$$\sum_{n=1}^{\infty} b^{3\ln n} \text{ is convergent } \Leftrightarrow \ln\left(b^{-3}\right) > 1 \qquad (1\text{pt})$$

$$\Leftrightarrow 0 < b < e^{-1/3}. \qquad (1\text{pt})$$

Consider

$$\lim_{n\to\infty} \frac{b^{\ln(n^3+1)}}{b^{\ln n^3}} = b^{\ln\left(1+\frac{1}{n^3}\right)} = b^0 = 1. \qquad (1\text{pt})$$

From the result and by the limit comparison test,

$$\sum_{n=1}^{\infty} b^{\ln(n^3+1)} \text{ is convergent } \Leftrightarrow \sum_{n=1}^{\infty} b^{\ln(n^3)} \text{ is convergent } \Leftrightarrow 0 < b < e^{-1/3}.$$