一百一十一學年度微積分甲 (二) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 9 頁。

(3) 試卷包括選擇題與計算/證明題，總分共計 100 分，占學期成績之 30%。
考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則
不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Which one of the following series is conditionally convergent?

   (A) $\sum_{n=1}^{\infty}(-1)^n \ln(1+\frac{1}{n^2})$.

   (B) $\sum_{n=100}^{\infty}(-1)^n \sin(\frac{\pi^2}{n^2})$.

   (C) $\sum_{n=100}^{\infty}(-1)^n \sin(\frac{\pi}{n})$.

   (D) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$.

2. Let $f(x,y)$ be a continuous function defined for $-\infty < x, y < \infty$ that satisfies

$$\lim_{(x,y)\to(\sqrt{2},-\sqrt{2})} \frac{f(x,y)+\sqrt{2}-\sqrt{2}(x-\sqrt{2})-2\sqrt{2}(y+\sqrt{2})}{\sqrt{(x-\sqrt{2})^2+(y+\sqrt{2})^2}} = 0.$$

   Then, $f(\sqrt{2},-\sqrt{2})+f_x(\sqrt{2},-\sqrt{2})+f_y(\sqrt{2},-\sqrt{2})$ is equal to

   (A) $\sqrt{2}$. (B) $2\sqrt{2}$. (C) $3\sqrt{2}$. (D) $4\sqrt{2}$.

3. Let $\mathbf{u}$ be a unit vector. Suppose that $f(x, y)$ is defined on an open disk D containing the point $(0, 0)$, and $f_x(x, y)$ and $f_y(x, y)$ both exist on whole D. Which one of the following statements must be **TRUE**?

   (A) If $f(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \varepsilon_1 \cdot x + \varepsilon_2 \cdot y$, then $f(x, y)$ is differentiable at $(0, 0)$.

   (B) If $f_x(x, y)$ and $f_y(x, y)$ are continuous at $(0, 0)$, then $f(x, y)$ is differentiable at $(0, 0)$.

   (C) If $f_{xy}(0, 0), f_{yx}(0, 0)$ both exist, then $f_{xy}(0, 0) = f_{yx}(0, 0)$.

   (D) If $D_{\mathbf{u}}f(0, 0)$ exists for any $\mathbf{u}$, then $D_{\mathbf{u}}f(0, 0) = \langle\, f_x(0, 0),\ f_y(0, 0)\,\rangle \cdot \mathbf{u}$.

4. If $u = xe^{ty}$, where $x = \alpha^2\beta$, $y = \beta^2\gamma$, and $t = \gamma^2\alpha$, find the value of $\dfrac{\partial u}{\partial \gamma}$ when $\alpha = -1$, $\beta = 2$, and $\gamma = 1$.

   (A) $8e^{-4}$. (B) $-8e^{-4}$. (C) $24e^{-4}$. (D) $-24e^{-4}$.

5. Let D be the region on the xy-plane inside the circle $x^2 + y^2 = 1$ and between lines $x + y = 1$ and $x + y = -1$. Then, $\iint\limits_D \sin(x - y)\, dA$ equals to

   (A) 0. (B) π. (C) $-\pi$. (D) 1.

6. Find the volume of the solid whose region is the intersection of the cube $\{(x, y, z) : |x| \le 1, |y| \le 1, |z| \le 1\}$ and the ball $\{(x, y, z) : x^2 + y^2 + z^2 \le 2\}$.

   (A) $32(\sqrt{2} - 1)$. (B) $(8 - \frac{16}{3}\sqrt{2})\pi$. (C) $(10 - \frac{16}{3}\sqrt{2})\pi$. (D) $\frac{32}{3}\sqrt{2}\pi$.

7. Consider the solid enclosed by the surface $(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2)$. Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$ be the spherical coordinate. Which one of the following options can correctly express the volume of the solid?

   (A) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\cos\phi\sin^2\phi} \rho^2 \sin\phi\, d\rho d\phi d\theta.$

   (B) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos^2\phi\sin\phi} \rho^2 \sin\phi\, d\rho d\phi d\theta.$

   (C) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\cos^2\phi\sin^2\phi} \rho^2 \sin\phi\, d\rho d\phi d\theta.$

   (D) $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi\sin^2\phi} \rho^2 \sin\phi\, d\rho d\phi d\theta.$

8. Consider the power series

$$\sum_{n=2}^{\infty} \frac{a^n (n!)^2}{(2n)! \ln n} x^n.$$

Which one of the following values of a that makes the radius of convergence of the power series equal to 2?

(A) -2. (B) 0. (C) 1/2. (D) 4.

9. Which one of the following equations represents the plane that is tangent to the two parametric curves $\gamma_1(s)$ and $\gamma_2(t)$ at their intersection point, where $\gamma_1(s) = \langle s+1, s^2+3, -s \rangle$ and $\gamma_2(t) = \langle t-2, t^2, -t+3 \rangle$.

(A) $y + z - 5 = 0$.
(B) $x + 2y + 3z = 6$.
(C) $x + z - 1 = 0$.
(D) $y + z - 1 = 0$.

10. Let C be the curve of intersection of the surface $z = \ln(x^2 + y^2 - 2)$ and the cylinder $(x-1)^2 + y^2 = 1$ in the first octant. Find the length of C.

(A) $\ln(1+\sqrt{2})$. (B) $\ln(1+\sqrt{3})$. (C) $\ln(2+\sqrt{2})$. (D) $\ln(2+\sqrt{3})$.

◎ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points, for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Let $f(x,y) = \sum_{n=1}^{\infty} \frac{(x^n + (2y)^n)^n}{n!}$. Which of the following statements are **TRUE**?

(A) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.
(B) $f_x(0,0) = 1$.
(C) $f_y(0,0) = 4$.
(D) f is differentiable at $(0,0)$.

12. Consider the following function

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^4}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Which of the following statements are **FALSE**?

(A) $f(x,y)$ is not continuous at $(0,0)$.

(B) $f_y(0,0)$ does not exist.

(C) $D_{\mathbf{u}}f(0,0)$ does not exist for some $\mathbf{u} = \langle\, a,\, b\,\rangle$ with $a \cdot b \neq 0$.

(D) $f_x(x,y)$ is not continuous at $(0,0)$.

13. Let $f(x,y) = x^2-2x$ and $g(x,y) = x^4-x^3+y^2$. Which of the following statements are **TRUE**?

(A) The function $g(x,y)$ has a local minimum value at $(x,y) = (3/4,0)$.

(B) The function $g(x,y)$ has no extreme value at $(x,y) = (0,0)$.

(C) Let $F(x,y) = [f(x,y)]^2/g(x,y)$. Then F has an absolute maximum value on D, where $D = \{(x,y) |\, (x-\frac{3}{2})^2 + y^2 < \frac{1}{4}\}$.

(D) The function $f(x,y)$ subject to the constraint $g(x,y) = 0$ has only one extreme value and it occurs at $(x,y) = (1,0)$.

14. Let f be a differentiable function of two variables. If the tangent plane to $z = f(x,y)$ at $(x,y) = (-3,4)$ is $z = 5x - 2y$. Which of the following planes must be tangent to $z = f(x^2 - y^2, 2xy)$?

(A) $z = 2x - 24y + 23$.

(B) $z = -2x + 24y - 69$.

(C) $z = 5x - 2y + 23$.

(D) $z = -5x + 2y - 23$.

15. Given a region Q enclosed by the surfaces $x^2 + y^2 + z^2 \leq 1$ and $z^2 \leq 3(x^2 + y^2)$. Which of the following options can correctly express the region Q?

(A) $Q = \{(x, y, z) : -\sqrt{3(x^2 + y^2)} \leq z \leq \sqrt{3(x^2 + y^2)}, 0 \leq x^2 + y^2 \leq \frac{1}{4}\} \cup \{(x, y, z) : -\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}, \frac{1}{4} \leq x^2 + y^2 \leq 1\}$.

(B) Let (r, θ, z) be the cylindrical coordinate. Then $Q = \{(r, \theta, z) : -\sqrt{1 - r^2} \leq z \leq \sqrt{1 - r^2}, 0 \leq r \leq \frac{1}{2}, 0 \leq \theta \leq 2\pi\} \cup \{(r, \theta, z) : -\sqrt{3}r \leq z \leq \sqrt{3}r, \frac{1}{2} \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$.

(C) Let (r, θ, z) be the cylindrical coordinate. Then $Q = \{(r, \theta, z) : -\sqrt{3}r \leq z \leq \sqrt{3}r, 0 \leq r \leq \frac{1}{2}, 0 \leq \theta \leq 2\pi\} \cup \{(r, \theta, z) : -\sqrt{1 - r^2} \leq z \leq \sqrt{1 - r^2}, \frac{1}{2} \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$.

(D) Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$ be the spherical coordinate. Then $Q = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, \frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}\}$.

⊙ **Part 3: 計算/證明題 (Questions of calculations and proofs)**
(兩個題組，共二十五分。)
(2 questions are worth for 25 points in total.)
(答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分。如果
(計算錯誤，則酌給部分分數。如果只有答案對，但觀念錯誤，或是過程不合理，
(則無法得到分數。)
(Answer the problems as thoroughly as possible. Be sure to include all your work. Partial credit will be given even if the answer is not fully correct.)

1. Let Ω be the xy-plane; it means $\Omega = \{(x,y) : -\infty < x, y < \infty\}$. Given a double integral
$$\iint_\Omega e^{-(x^2+2xy+5y^2)}\,dA.$$
   (A) (4 points) Let $u = x + y$ and $v = 2y$. Evaluate the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

   (B) (4 points) Use the change of variables from (A) to write down the integral form with variables u and v for
$$\iint_\Omega e^{-(x^2+2xy+5y^2)}\,dA.$$

   (C) (7 points) We have known the integral $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Use (A) and (B) to evaluate the value of the double integral
$$\iint_\Omega e^{-(x^2+2xy+5y^2)}\,dA.$$

2. Let $\{a_n\}$ be a sequence of positive numbers and σ be a positive number.

(A) (3 points) If there exists some positive integer N such that

$$(\ln n)^{-1} \cdot \ln\left(\frac{1}{a_n}\right) \geq 1 + \sigma \text{ holds for } n \geq N.$$

Prove that $\sum_{n=1}^{\infty} a_n$ is convergent.

(B) (3 points) If there exists some positive integer N such that

$$(\ln n)^{-1} \cdot \ln\left(\frac{1}{a_n}\right) \leq 1 \text{ holds for } n \geq N.$$

Prove that $\sum_{n=1}^{\infty} a_n$ is divergent.

(C) (4 points) By using the results in (A), (B) or otherwise, find all $b \in (0, \infty)$ such that $\sum_{n=1}^{\infty} b^{\ln(n^3+1)}$ is convergent.