一百一十二學年度微積分甲 (一) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 9 頁。

(3) 試卷包括選擇題、填充題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Let $f(x) = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + e^x$. How many real roots does $f(x)$ have in $(-\infty, \infty)$?

   (A) 0. (B) 1. (C) 2. (D) 3.

2. Suppose that g is differentiable on $(-\infty, \infty)$ with $g(0) > 0$, and g satisfies

   $$(g(x))^2 + (x^2+1)g(x) + 2x - 2 = 0 \quad \text{for all } x \in (-\infty, \infty).$$

   Find $g'(0)$.

   (A) -4. (B) $-\frac{2}{3}$. (C) $-\frac{2}{5}$. (D) 0.

3. The value of $\lim_{x\to 5} \frac{1}{x-5}\int_5^x \frac{x\sin t}{t}dt$ is

   (A) 0. (B) $\cos 5$. (C) $\sin 5$. (D) Does not exist.

4. Let $f(x) = e^{-x}(\sin x - \cos x)$. Find the **average value** of f on the interval $[0, 2\pi]$.

   (A) 0. (B) $\frac{1}{2\pi}(e^{-2\pi} - 1)$. (C) $\frac{1}{2\pi}e^{-2\pi}$. (D) $-\frac{1}{2\pi}e^{-2\pi}$.

5. Let $f(x) = \int_0^x \sec\theta d\theta$ and $g(x) = \int_{x^2}^0 f(t)dt$. Find $g'(\frac{\sqrt{\pi}}{2})$.

   (A) $-\sqrt{\pi}\ln(\sqrt{2}+1)$.

   (B) $-\ln(\sqrt{2}-1)$.

   (C) $\sqrt{\pi}\ln(\sqrt{2}+1)$.

   (D) $\ln(\sqrt{2}-1)$.

6. Let

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

   Find the **largest** number $\delta > 0$ such that if $0 < |x| < \delta$, we have $|f(x) - 1| < 1$.

   (A) $\frac{\pi}{2}$. (B) $\frac{3\pi}{4}$. (C) π. (D) 2π.

7. Assume that $f(x)$ is differentiable on $(-\infty, \infty)$ and satisfies $0 \le f'(x) \le 4$ for all $x \in [0, 2]$ and $\int_0^2 f(t)\,dt = \frac{8}{3}$. Which of the following **cannot** be a possible value of $f(0)$?

   (A) 0. (B) 1. (C) -1. (D) 2.

8. Let $R = \{(x, y) | \, x \ge 1, \ 0 \le y \le \frac{1}{x}\}$. By rotating R about the x-axis we obtain a resulting solid S. Which of the following statements is **TRUE**?

   (A) The volume of S is finite, and the surface area of S is finite.

   (B) The volume of S is finite, and the surface area of S is infinite.

   (C) The volume of S is infinite, and the surface area of S is finite.

   (D) The volume of S is infinite, and the surface area of S is infinite.

9. Let f be a function defined on $(-1, 2)$. Which of the following statements must be **TRUE**?

    (A) If f is continuous on $(-1, 2)$, then f is differentiable on $(-1, 2)$.

    (B) If $f''(x)$ exists and $f''(x) < 0$ for all $x \in (-1, 2)$, then

    $$f(\theta) \leq (1-\theta)f(0) + \theta f(1) \quad \text{for all } \theta \in (0, 1).$$

    (C) If f is not continuous at 0, then f cannot attain an absolute maximum value on $(-1, 2)$.

    (D) If $f'(x)$ exists and $f'(x) > 0$ for all $x \in (-1, 2)$, then $f(x) < f(y)$ for all $-1 < x < y < 2$.

10. Consider the curve C parametrized by $x = 8\cos^3\theta$ and $y = 8\sin^3\theta$ for $0 \leq \theta \leq 2\pi$. Which of the following statements is **NOT TRUE**?

    (A) C is symmetric with respect to x-axis.

    (B) C can be expressed by the Cartesian equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$.

    (C) There are precisely two points on C where the tangent lines have a slope of 1.

    (D) The area enclosed by C is 24.

⊙ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points, for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Let f and g be functions defined on $(-\infty, \infty)$. Which of the following must be **TRUE**?

(A) If $\lim_{x\to 0} f(x) = A$ for some real number A, and $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0}(f(x) + g(x))$ does not exist.

(B) If $\lim_{x\to 0} f(x) = A$ for some real number A, and $\lim_{x\to 0} g(x)$ does not exist, then $\lim_{x\to 0}(f(x)g(x))$ does not exist.

(C) If $\lim_{x\to 0}(f(x) + g(x)) = A$ and $\lim_{x\to 0}(f(x) - g(x)) = B$ for some real numbers A and B, then $\lim_{x\to 0}(f(x)g(x))$ exists.

(D) If f and g are differentiable in $(-\infty, \infty)$ and $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = A$ for some real number A, then $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = A$.

12. Let $y = f(x) = cx + \ln\cos x$, where c is a real number. Which of the following are **TRUE**?

(A) $y = f(x)$ is one-to-one on $(-\frac{\pi}{2}, \frac{\pi}{2})$ for any real number c.

(B) $y = f(x)$ is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ for any real number c.

(C) $y = f(x)$ is integrable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$ for any real number c.

(D) The linear approximation of $y = f(x)$ at $x = 0$ is $y = cx$.

13. Let $y = f(x) = \dfrac{1}{x^2 + 2x + c}$, where c is a real number. Which of the following are **TRUE**?

(A) $y = f(x)$ has no vertical asymptote when $c > 1$.

(B) $y = f(x)$ has two vertical asymptotes when $c \leq 1$.

(C) $y = f(x)$ has a local maximum at $x = -1$ when $c \leq 1$.

(D) $y = f(x)$ has an absolute maximum at $x = -1$ when $c > 1$.

14. Which of the following integrals are convergent?

(A) $\displaystyle\int_0^2 \frac{1}{x^2 - 8x + 7} dx.$

(B) $\displaystyle\int_2^\infty \frac{1}{x^2 \ln(1 + x^3)} dx.$

(C) $\displaystyle\int_0^1 x^{\frac{1}{2}} \ln x dx.$

(D) $\displaystyle\int_{-\infty}^\infty \tan^{-1} x dx.$

15. Let f be a function satisfying $f(x + y) = f(x)f(y)$ and $f(x) > 0$ for any $x, y \in (-\infty, \infty)$. Which of the following must be **TRUE**?

(A) $f(0) = 1$.

(B) $f(112x) = (f(x))^{112}$ for all $x \in (-\infty, \infty)$.

(C) If $f'(0)$ exists, then f is differentiable on $(-\infty, \infty)$.

(D) If f is continuous at some real number a, then f is continuous on $(-\infty, \infty)$.

⊙ **Part 3: 填充題與計算/證明題**
(Fill-in-the-Blank Questions, and Questions of calculations and proofs)
(兩個題組，共二十五分。第一題為填充題，第二題為計算證明題。)
(There are two questions worth a total of 25 points. The first question is a fill-in-the-blank question, and the second one is a question of calculations and proofs.)
(計算/證明題答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分；如果計算錯誤，則酌給部分分數；如果只有答案對，但觀念錯誤，或是過程不合理，則無法得到分數。)
(Answer the question of calculations and proofs as thoroughly as possible. In the case of computational errors, partial credit may be given. However, if only the answer is correct but there are conceptual errors or an unreasonable process, no credit will be awarded.)

1. Given the region E enclosed by these curves $y = x^2 - \dfrac{\ln x}{8}$, $x = 1$, $x = t$ and $y = 0$, where $t > 1$ is a real number. Denote $A(t)$ as the area of the region E, and $L(t)$ as the perimeter (周長) of E, i.e. the total length of the boundary (outer edge) of E.

   (A) (5 pts) If
   $$L(t) = at^2 + bt - 1$$
   for some real numbers a and b, then $(a, b) =$ <u>(1)</u>.

   (B) (5 pts) If
   $$A(t) = pt^3 + q(t - t\ln t) + r$$
   for some real numbers p, q and r, then $(p, q, r) =$ <u>(2)</u>.

   (C) (5 pts) There exist real numbers α and $\beta > 0$ such that $\displaystyle\lim_{t\to\infty} \frac{A(t)}{t^\alpha L(t)} = \beta$. Then $(\alpha, \beta) =$ <u>(3)</u>.

2. Let $f(x)$ be a differentiable function, and its derivative $f'(x)$ be continuous in $[0,1]$. Let n be a positive integer.

   (A) (3 pts) State the Mean Value Theorem for the function f on the closed interval $[\frac{1}{2n}, \frac{2}{2n}]$.

   (B) (3 pts) Prove that there exists a $c_j \in (\frac{2j-1}{2n}, \frac{2j}{2n})$, $j = 1, 2, \cdots, n$, such that

$$\sum_{k=1}^{2n} (-1)^k f(\frac{k}{2n}) = \frac{1}{2n} \sum_{j=1}^{n} f'(c_j).$$

   (C) (4 pts) Compute

$$\lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^k \, e^{\cos \frac{k}{2n}\pi}.$$