一百一十二學年度微積分甲 (二) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 9 頁。

(3) 試卷包括選擇題、填充題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Let $z = xe^{\frac{y}{x}}$. Find $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$ at $(x, y) = (3, 6)$.

   (A) $2e^2$; (B) $3e^2$; (C) $4e^2$; (D) $9e^2$.

2. In each of the following choices, each function is matched with the interval of convergence of its Maclaurin series. Which matching is **FALSE**?

   (A) $\dfrac{1}{2-x}$ and $-2 < x < 2$;

   (B) $e^{\frac{x^2}{2}}$ and $-\infty < x < \infty$;

   (C) $\ln(4 + x)$ and $-4 \le x \le 4$;

   (D) $\tan^{-1}(\dfrac{x}{3})$ and $-3 \le x \le 3$.

3. Find the surface area of the part of the surface $2y + 4z - x^2 - 10 = 0$ that lies above the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$ on the xy-plane.

   (A) $9 - \dfrac{5}{3}\sqrt{5}$; (B) $7 - \dfrac{2}{3}\sqrt{5}$; (C) $5 + \dfrac{5}{3}\sqrt{5}$; (D) $\dfrac{2}{3}\sqrt{2}$.

4. For $a > b > 0$, given a series $\displaystyle\sum_{n=1}^{\infty}(\frac{a^n}{n} + \frac{b^n}{n^2})x^n$. The exact interval of convergence of this series is

   (A) $[-\dfrac{1}{b}, \dfrac{1}{b}]$; (B) $(-\dfrac{1}{a}, \dfrac{1}{a})$; (C) $(-\dfrac{1}{b}, \dfrac{1}{b}]$; (D) $[-\dfrac{1}{a}, \dfrac{1}{a})$.

5. Which of the following plane passing through $(0,-2,-1)$ and $(2,1,1)$ is tangent to the paraboloid $z=x^2+y^2$?

(A) $4x-2y-z=5$;

(B) $4x+6y-11z=-1$;

(C) $x+2y-4z=0$;

(D) $x-4y+5z=3$.

6. Using the transformation $x=u^2-v^2$ and $y=2uv$, the integral

$$\int_0^1\int_0^{2\sqrt{1-x}}\sqrt{x^2+y^2}dydx$$

equals

(A) $4\int_0^1\int_0^u(u^2+v^2)^2dvdu$;

(B) $4\int_0^1\int_0^1(u^2+v^2)^2dvdu$;

(C) $2\int_0^1\int_1^u(u^2+v^2)^2dvdu$;

(D) $\int_0^1\int_0^1(u^2+v^2)dvdu$.

7. Find the volume of the solid enclosed by the surface $z=\ln(x^2+y^2)$, the plane $z=0$, the cylindrical surfaces $x^2+y^2=1$ and $x^2+y^2=4$.

(A) $(3+4\ln 2)\pi$; (B) $(-3+4\ln 2)\pi$; (C) $(3+4\ln 4)\pi$; (D) $(-3+4\ln 4)\pi$.

8. Let

$$f(x,y)=\begin{cases}\dfrac{xy^3-x^3y}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

Which of the following is **TRUE**?

(A) f is not continuous at $(0,0)$;

(B) f is differentiable at $(0,0)$;

(C) $f_{xy}(0,0)=f_{yx}(0,0)$;

(D) For the rectangle $R=\{(x,y)\colon -1\le x\le 1, -2\le y\le 2\}$,

$$\int_{-2}^2\int_{-1}^1 f(x,y)\,dxdy\neq\int_{-1}^1\int_{-2}^2 f(x,y)\,dydx.$$

9. Consider the region $D = \left\{(x, y) : x^2 + y^2 \leq \frac{3}{16}\right\}$. The integral

$$\iint_D \min\left\{\left(\frac{3}{16} - x^2 - y^2\right)^{\frac{1}{2}}, 2(x^2 + y^2)\right\} dA$$

equals

(A) $\frac{5}{194}\pi$; (B) $\frac{5}{193}\pi$; (C) $\frac{5}{192}\pi$; (D) $\frac{5}{191}\pi$.

10. The value of

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+3)} \frac{1}{3^{k+1}}$$

equals

(A) $\frac{\pi\sqrt{3} - 9}{18}$; (B) $\frac{2\pi\sqrt{3} - 9}{18}$; (C) $\frac{2\pi\sqrt{3} - 6}{18}$; (D) $\frac{2\pi\sqrt{3} - 9}{6}$.

◎ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points, for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Which of the following series are convergent?

(A) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+7}$;

(B) $\sum_{n=1}^{\infty} \frac{\sin(2n+5)}{2^n - n}$;

(C) $\sum_{n=1}^{\infty} n^2 e^{-n}$;

(D) $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{1}{n^5})$.

12. Let $f(x,y) = \displaystyle\int_{x-y}^{x^2+y} \cos(t^2)\,dt$ and $F(x) = \displaystyle\int_0^x \cos(t^2)\,dt$. Which of the following statements are **TRUE**?

(A) $f(x,y) = F(x^2+y) - F(x-y)$;

(B) $f_x(x,y) = 2x\cos\left((x^2+y)^2\right) - \cos\left((x-y)^2\right)$;

(C) $f_{xy}(0,0) = 0$;

(D) The maximum value of the directional derivative of f at $(0,0)$ is $\sqrt{5}$.

13. Which of the following statements are **TRUE**?

(A) If f is continuous in $[0,1]$, then

$$\int_0^1\int_0^1\int_0^1 f(x)f(y)f(z)dxdydz = \left(\int_0^1 f(x)dx\right)^3;$$

(B) The Jacobian of the transformation $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$ and $z = r\cos\theta$ is $r^2\sin\theta$, where $r > 0$, $0 \le \phi \le \frac{\pi}{3}$ and $0 \le \theta \le \frac{\pi}{2}$;

(C) The transformation $x = r\cos\theta$ and $y = r\sin\theta$ for $r > 0$ and $0 \le \theta \le \frac{9\pi}{2}$ is one-to-one;

(D) For $a, b, c > 0$, the volume of $D = \left\{(x,y,z)\colon \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 4\right\}$ is $\frac{32\pi}{3}abc$.

14. Which of the following statements must be **TRUE**.

(A) There exists a real-valued function f of two variables such that both $D_{<\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}>}f(0,0)$ and $D_{<\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}>}f(0,0)$ exist, yet $D_{\mathbf{u}}f(0,0)$ does **NOT** exist for any other unit vectors $\mathbf{u}$;

(B) If $f_x(x,y) = 0$ for all $(x,y) \in D$, where D is a disk, then f is a constant function on D;

(C) For differentiable functions $f(x,y)$ and $g(x,y)$, we take

$$L(x,y) = f(x,y) - \lambda g(x,y), \quad -\infty < \lambda < \infty.$$

If $L_x(0,0) = L_y(0,0) = 0$, $\lambda \neq 0$ and $\nabla g(0,0) \neq <0,0>$, then $\nabla f(0,0)$ and $\nabla g(0,0)$ are parallel.

(D) Let

$$z = f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

and $x(t) = y(t) = t$ for $-\infty < t < \infty$. Then $\left.\dfrac{dz}{dt}\right|_{t=0} = 0$.

15. Consider the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where

$$x(t) = t^2 + t + 4, \quad y(t) = \frac{t}{t+1}, \quad z(t) = e^{t^2}. \tag{1}$$

Let C be a curve with parametric equations (1). Which of the following are **TRUE**?

(A) The vector function $\mathbf{r}(t)$ is continuous at $t = -1$;

(B) The unit tangent vector of $\mathbf{r}(t)$ at $t = 0$ is $\dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$;

(C) The arc length of C from the point $(4,0,1)$ to the point $(6,\frac{1}{2},e)$ is equal to

$$\int_0^1 \sqrt{1 + 4(t^2 + t + t^2 e^{2t^2}) + \frac{1}{(1+t)^4}}\, dt;$$

(D) $\dfrac{d}{dt}|\mathbf{r}'(t)| = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$, where $\mathbf{r}'(t)$ means the derivative of $\mathbf{r}(t)$ with respect to t.

◎ **Part 3: 填充題與計算/證明題**
(Fill-in-the-Blank Questions, and Questions of calculations and proofs)
(兩個題組，共二十五分。第一題為填充題，第二題為計算證明題。)
(There are two questions worth a total of 25 points. The first question is a fill-in-the-blank question, and the second one is a question of calculations and proofs.)
(計算/證明題答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分；如果計算錯誤，則酌給部分分數；如果只有答案對，但觀念錯誤，或是過程不合理，則無法得到分數。)
(Answer the question of calculations and proofs as thoroughly as possible. In the case of computational errors, partial credit may be given. However, if only the answer is correct but there are conceptual errors or an unreasonable process, no credit will be awarded.)

1. Consider the function $f(x, y) = x^2 - xy + y^4$.

   (A) (5 pts) Find all critical points of f.

   (B) (5 pts) Find all points (x, y) where f has a local minimum.

   (C) (5 pts) Find the maximum value M and the minimum value m of f subject to the constraint $x^2 + y^4 = 3$.

2. Consider the region $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9\}$, and define the integral

$$I = \iiint_E \frac{z - 5}{\left(x^2 + y^2 + (z - 5)^2\right)^{\frac{3}{2}}} dV.$$

(A) (5 pts) Under spherical coordinate transformation, prove the following expression

$$I = 2\pi \int_0^3 \rho^2 \Big(\int_0^\pi f(\rho, \varphi) d\varphi \Big) d\rho,$$

where

$$f(\rho, \varphi) = \frac{(\rho \cos \varphi - 5) \sin \varphi}{(\rho^2 + 25 - 10\rho \cos \varphi)^{\frac{3}{2}}}, \quad 0 < \rho \leq 3.$$

(B) (3 pts) Define $G(\rho) = \int_0^\pi f(\rho, \varphi) d\varphi$ for $0 < \rho \leq 3$. Prove that $G(\rho)$ is a negative constant over the interval $(0, 3]$. Moreover, find the exact value of this negative constant.

Hint: You may use the substitution $t = (\rho^2 + 25 - 10\rho \cos \varphi)^{\frac{1}{2}}$.

(C) (2 pts) Find the exact value of I.