一百一十三學年度微積分甲 (一) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

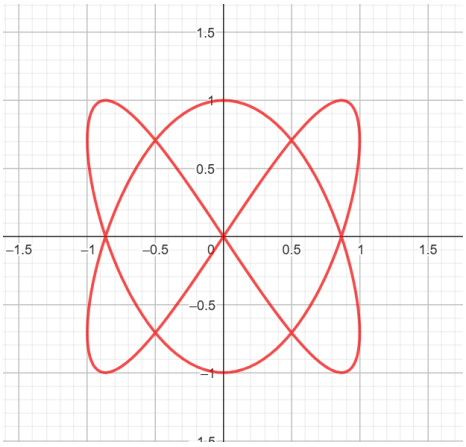
(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 6 頁。

(3) 試卷包括選擇題、填充題或計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Which pair of parametric equations represents the graph below?
(All options have the range of θ as $[0, 2\pi]$.)



(A) $\begin{cases} x = \theta - \sin(\theta) \\ y = \cos(\theta) \end{cases}$;

(B) $\begin{cases} x = \cos(\theta) \\ y = \theta + \sin(\theta) \end{cases}$;

(C) $\begin{cases} x = \cos(\theta) \\ y = \cos(4\theta) \end{cases}$;

(D) $\begin{cases} x = \sin(2\theta) \\ y = \cos(3\theta) \end{cases}$.

2. Find the curve symmetric to $y = e^x - e^{-x}$ about the line $y = -x$.

(A) $y = e^{-x} - e^x$; (B) $y = e^x - e^{-x}$; (C) $x = e^y - e^{-y}$; (D) $x = e^{-y} - e^y$.

3. Find the derivative of $f(x) = \tan^{-1}(\sec x + \tan x)$.

(A) $(\sec^{-1}(\sec x + \tan x))^2$; (B) $(1 + (\sec x + \tan x)^2)^{-1}$; (C) $\frac{1}{2}$; (D) $\frac{2}{5}$.

4. Let $f(t)$ be the average value of $1 + \sin^{99} x$ on $[0, t]$. Find the limit $\lim_{t\to\infty} f(t)$.

(A) $\frac{1}{2}$; (B) 1; (C) $\frac{3}{2}$; (D) Does not exist.

5. Let R be the region enclosed by the function $f(x) = x^3 - 2x^2 + 4$ and the horizontal line $y = 4$. Find the volume of the solid generated by rotating R about the horizontal line $y = -1$.

   (A) $\frac{193}{21}\pi$; (B) $\frac{992}{105}\pi$; (C) $\frac{424}{35}\pi$; (D) $\frac{451}{35}\pi$.

6. Find the derivative of $f(x) = \int_0^x \sin((x-t)^2)\, dt$.

   (A) $\sin(x^2)$; (B) $-\sin(x^2)$; (C) 0; (D) $2\cos(x^2)$.

7. Find the limit $\lim_{n\to\infty} \left\{ \frac{\sin\frac{\pi}{n}}{n+3} + \frac{\sin\frac{2\pi}{n}}{n+3} + \cdots + \frac{\sin\pi}{n+3} \right\}$.

   (A) $\frac{\pi}{2}$; (B) $\frac{2}{\pi}$; (C) π; (D) Does not exist.

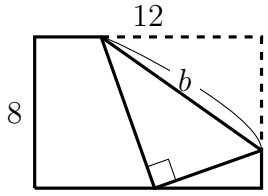
8. Consider the parametric curve $\gamma(t) = (r\sin(t), r\cos(t) + r^2)$ where $r > 0$ is a constant. Find the area of the surface obtained by rotating the curve about the x-axis from $t = 0$ to $t = \frac{\pi}{2}$.

   (A) $\pi r^2 + \pi r^3$; (B) $\pi r^2 + \pi^2 r^3$; (C) $2\pi r^2 + \pi r^3$; (D) $2\pi r^2 + \pi^2 r^3$.

9. Find the definite integral $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^3}{\sqrt{1-x^2}} dx$.

   (A) $\frac{2}{3} - \frac{5}{6\sqrt{2}}$; (B) $\frac{2}{3} + \frac{5}{6\sqrt{2}}$; (C) $\frac{1}{3} - \frac{1}{6\sqrt{2}}$; (D) $\frac{1}{3} + \frac{1}{6\sqrt{2}}$.

10. The upper right-hand corner of a piece of a paper, 12 cm by 8 cm, as in the figure, is folded over to the bottom. Find the minimum length of the fold (b in the figure).



   (A) $6\sqrt{3}$; (B) $6\sqrt{2}$; (C) $5\sqrt{5}$; (D) $8\sqrt{2}$.

◎ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。錯一個選項扣兩分，錯兩個選項以上不給分，分數不倒扣。)
(5 questions, each question is worth 5 points, for 25 points in total. The correct answer is worth 5 points. Answers at a distance 1 from the correct answer are worth 3 points, other answers are worth no points.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Let f be a one-to-one function satisfying that $f'''(x)$ exists, $f'(x) \neq 0$ and $f(x) > 0$ for all x. Let g be a function satisfying that $g'''(x)$ exists for all x.
Which of the following formulae must be **FALSE**?

(A) $(f(x)g(x))''' = f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + g'''(x)$;

(B) $(f(g(x)))'' = f''(g(x))g'(x) + f'(g(x))g''(x)$;

(C) $(f(x)^{g(x)})' = g(x)f(x)^{g(x)}f'(x) + f(x)^{g(x)}g'(x)\ln(f(x))$;

(D) $(f^{-1})'(x) = (f'(f^{-1}(x)))^{-1}$.

12. Let f be a function concave upward on (a, b) and $c \in (a, b)$.
Which of the following statements must be **TRUE**?

(A) $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$ exists; (B) $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$ exists;
(C) f is continuous at c; (D) f' is differentiable at c.

13. Let f be a function on $(-\infty, \infty)$ with a continuous first derivative.
Which of the following statements must be **TRUE**?

(A) If f is an odd function, then $\int_a^x f(t)\,dt$ is an even function for all a;

(B) If f is an even function, then $\int_a^x f(t)\,dt$ is an odd function for all a;

(C) If f is an odd function, then $\int_0^x (\cos(f'(t)) + f(t))\,dt$ is an odd function;

(D) If f is an odd function, then $\int_0^x (\cos(f(t)) + f'(t))\,dt$ is an odd function.

14. Consider the functions $f_1(x) = x^3$ and $f_2(x) = e^{2x}$. Let L_i be the arc length of $f_i(x)$ from $x = 0$ to $x = 1$, and let A_i be the area under $f_i(x)$ from $x = 0$ to $x = 1$, for $i = 1, 2$. Which of the following statements are **TRUE**?

(A) $L_1 > L_2$; (B) $A_1 > A_2$; (C) $L_1 > A_1$; (D) $L_2 > A_2$.

15. Find the extreme values of $\left(\frac{x}{x+y}\right)^{\sqrt{2}} + \left(\frac{y}{x+y}\right)^{\sqrt{2}}$ over the region $\Gamma = \{(x, y) : x, y \geq 0 \text{ and } x + y > 0\}$.

(A) 1; (B) $2^{-\sqrt{2}}$; (C) 2; (D) $2^{1-\sqrt{2}}$.

◎ **Part 3: 填充或計算/證明題**
(Fill-in-the-Blank Questions, or Questions of calculations and proofs)
(三個題組，共二十五分。前兩題為填充題，第三題為計算證明題。)
(There are three questions worth a total of 25 points. The first two questions are fill-in-the-block questions, and the third one is a question of calculations and proofs.)
(計算/證明題答題時應將推理或解題過程說明清楚，且得到正確答案，方可得到滿分；如果計算錯誤，則酌給部分分數；如果只有答案對，但觀念錯誤，或是過程不合理，則無法得到分數。)
(Answer the question of calculations and proofs as thoroughly as possibly. In the case of computational errors, partial credit maybe given. However, if only the answer is correct but there are conceptual errors of an unreasonable process, no credit will be awarded.

1. (7 pts) Let s and t be real numbers. Determine all values of $s-t$ such that $\displaystyle\int_0^{\pi/2} \frac{\sin^s x}{x^t}\,dx$ converges.

2. Let a be a real number with $a>1$.

   (A) (5 pts) Find the absolute minimum value of $f(x)=a^x x^a$ on $(1,\infty)$.

   (B) (5 pts) Determine all values of a such that $x^a \le a^x$ for all $x>1$.

3. Let f be a function continuous at 0 and assume that the following limit exists:

$$\lim_{x\to 0}\left(\frac{f(x)}{x^3}-\frac{3}{x^3}-\frac{2}{x^2}-\frac{1}{x}\right).$$

   (A) (2 pts) Find the value of $f(0)$.

   (B) (2 pts) Find the value of $f'(0)$.

   (C) (4 pts) Assume that $f''(0)$ exists. Find the value of $f''(0)$.