《一百一十三學年度第二學期微積分會考答案卷》(A、B 卷)

--------------------------------A 卷選擇題--------------------------------

第一部份：單選擇題

1	**C**	**2**	**A**	**3**	**B**	**4**	**D**	**5**	**B**
6	**D**	**7**	**A**	**8**	**A**	**9**	**B**	**10**	**D**

第二部份：複選擇題

11	**ABD**	**12**	**ACD**	**13**	**BCD**	**14**	**ABD**	**15**	**ABCD**

--------------------------------B 卷選擇題--------------------------------

第一部份：單選擇題

1	**A**	**2**	**B**	**3**	**C**	**4**	**D**	**5**	**D**
6	**B**	**7**	**A**	**8**	**A**	**9**	**D**	**10**	**B**

第二部份：複選擇題

11	**ACD**	**12**	**ABD**	**13**	**ABCD**	**14**	**BCD**	**15**	**ABD**

第三部份：填充題

1		**2**
(1)	**(2)**	**(3)**
$\mathbf{4}$	$\mathbf{2(\pi - \tan^{-1} 2)}$	$\boldsymbol{\frac{q}{1-q}}$

一百一十三學年度微積分甲 (二) 會考計算證明題評分

1. Let D be the region on the xy-plane enclosed by the curve

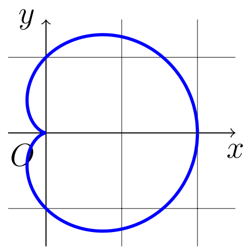
$$S : (x^2 + y^2 - x)^2 = x^2 + y^2.$$

(A) (2 pts) Show that $r = \cos\theta - 1$ is a polar equation for S.

(B) (4 pts) Find the area of D.

(C) (4 pts) Find the average value of $f(x, y) = \sqrt{x^2 + y^2}$ over D.

Solution:



$r = \cos\theta \pm 1$

(A) (座標變換, 代入檢驗, 各一分)

Let $\boxed{x = r\cos\theta,\ y = r\sin\theta}$. (1pt)

Then $x^2 + y^2 = r^2 = r(\cos\theta - 1)$ and thus
$(x^2 + y^2 - x)^2 = (r(\cos\theta - 1) - r\cos\theta)^2$
$= (-r)^2 = r^2 = x^2 + y^2$. (1pt)

.... Or ....

$(x^2 + y^2 - x)^2 = ((\cos\theta - 1)^2 - (\cos\theta - 1)\cos\theta)^2 = (-(\cos\theta - 1))^2$
$= (\cos\theta - 1)^2 = x^2 + y^2$. (1pt)

.... Or ....

$(r^2 - r\cos\theta)^2 = (x^2 + y^2 - x)^2 = x^2 + y^2 = r^2,$
$\implies r = 0$ or $(r - \cos\theta)^2 = 1$, $r = \cos\theta \pm 1$. (the same curve) (1pt)

(B) (面積公式, 定積分/迭代積分: 各一分; 面積值: 兩分)

$r = \cos\theta - 1 \iff -r = \cos(\theta + \pi) - 1 = -\cos\theta - 1 \iff r = 1 + \cos\theta,$
$D = \{(r, \theta) : 0 \le r \le 1 + \cos\theta,\ 0 \le \theta \le 2\pi\}.$

$$A(D) \overset{(1pt)}{=} \left\{ \begin{array}{l} \boxed{\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta} \overset{(1pt)}{=} \boxed{\int_0^{2\pi} \frac{1}{2}(1 + \cos\theta)^2 \, d\theta} \\ \text{.......................... Or ....} \\ \boxed{\iint_D dA} \overset{(1pt)}{=} \boxed{\int_0^{2\pi} \int_0^{1+\cos\theta} r \, dr \, d\theta} \\ = \int_0^{2\pi} \left[\frac{1}{2} r^2\right]_0^{1+\cos\theta} d\theta = \int_0^{2\pi} \frac{1}{2}(1 + \cos\theta)^2 \, d\theta \end{array} \right\}$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \cos\theta + \frac{1 + \cos 2\theta}{4}\right) d\theta = \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{8}\right]_0^{2\pi} = \boxed{\frac{3}{2}\pi}. \text{ (2pts)}$$

(C) (迭代積分, 值, 平均值公式, 平均值: 各一分)

$$\iint_D f(x,y)\,dA = \iint_D \sqrt{x^2+y^2}\,dA \overset{(1pt)}{=} \boxed{\int_0^{2\pi}\int_0^{1+\cos\theta} r\cdot r\,dr\,d\theta}$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^{1+\cos\theta} d\theta = \int_0^{2\pi} \frac{1}{3}(1+\cos\theta)^3\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{5}{6} + \frac{4}{3}\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{3}\sin^2\theta\cos\theta\right)d\theta$$

$$= \left[\frac{5}{6}\theta + \frac{4}{3}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{9}\sin^3\theta\right]_0^{2\pi} = \boxed{\frac{5}{3}\pi}, \text{ (1pt)}$$

$$f_{\text{ave}} \overset{(1pt)}{=} \boxed{\frac{1}{A(D)}\iint_D f(x,y)\,dA} = \frac{5}{3}\pi \div \frac{3}{2}\pi = \boxed{\frac{10}{9}}. \text{ (1pt)}$$