一百一十三學年度微積分甲 (二) 會考試題 (A 卷)

說明：

(1) 答題之前請先檢查所取得之試卷與答案卷、卡卷別是否相符。

(2) 測驗時間 120 分鐘。試卷加答案卷、答案卡共計 7 頁。

(3) 試卷包括選擇題、填充題與計算/證明題，總分共計 100 分，占學期成績之 30%。考卷成績將作為微積分獎給獎依據。

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答，否則不予計分。

◎ **Part 1: 單選擇題**
(Multiple-Choice Questions, Each Problem with Single Correct Answer)
(單選十題，每題五分，共五十分。)
(10 questions, each question is worth 5 points, for 50 points in total.)

1. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-2)^n}{n^2+2} x^{2n+1}$.

   (A) $[-2, 2]$; (B) $(-2, 2)$; (C) $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$; (D) $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

2. Let a space curve γ be given by the parametric equations:

$$x = \int_0^t \cos(1+4u^2)\, du, \quad y = \int_0^t \sin(1+4u^2)\, du, \quad 0 \le t \le \pi.$$

   Then the length of γ is

   (A) π; (B) 2π; (C) 1; (D) 2.

3. Suppose z is a differentiable function of u and v, and u and v are differentiable functions of x and y. When $(x, y) = (x_0, y_0)$, we have $\frac{\partial z}{\partial u} = 5$, $\frac{\partial z}{\partial v} = 2$, $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = 3$, $\frac{\partial v}{\partial x} = -2$, and $\frac{\partial z}{\partial y} = 7$. Then the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ at (x_0, y_0) is

   (A) -2; (B) 2; (C) -4; (D) 4.

4. The Maclaurin series for $x \cos\left(\frac{x^2}{2}\right)$ is

   (A) $\frac{1}{4} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; (B) $\frac{1}{4} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$;

   (C) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2^n (2n)!}$; (D) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4^n (2n)!}$.

5. Let E be the solid enclosed by the surface $|x|+|y|+|z|=1$. Find the value of $\iiint\limits_E (x+y)ze^{x^2+y^2+z^2}\,dV$.

(A) $-e$; (B) 0; (C) 1; (D) e.

6. Which of the following series does diverges?

(A) $\sum\limits_{n=0}^{\infty} n\sin\left(\frac{\pi}{2^n}\right)$; (B) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$;

(C) $\sum\limits_{n=0}^{\infty} \left(\frac{7}{3}\right)^n \left(\frac{n}{n+1}\right)^{n^2}$; (D) $\sum\limits_{n=1}^{\infty} \frac{\sin^n 1}{\cos^n 1}$.

7. Suppose f is a differentiable function. Let $w=f(x-at)$, $x=u-v$, $t=v-u$, and a be a constant. When $u=1$ and $v=0$, $\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}=$

(A) 0; (B) $af'(1+a)$; (C) $-af'(1+a)$; (D) $f'(1+a)$.

8. Let

$$\begin{aligned} u &= 1+\frac{(2x)^3}{3!}+\frac{(2x)^6}{6!}+\frac{(2x)^9}{9!}+\cdots, \\ v &= 2x+\frac{(2x)^4}{4!}+\frac{(2x)^7}{7!}+\frac{(2x)^{10}}{10!}+\cdots, \\ w &= \frac{(2x)^2}{2!}+\frac{(2x)^5}{5!}+\frac{(2x)^8}{8!}+\cdots. \end{aligned}$$

Then $u^3+v^3+w^3-3uvw=$

(A) 1; (B) $1+x$; (C) $1-x$; (D) e^x.

9. Find the value of $\iiint\limits_E (x^2+y^2)\,dV$, where E lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

(A) $\frac{232}{15}\pi$; (B) $\frac{248}{15}\pi$; (C) $\frac{256}{15}\pi$; (D) $\frac{272}{15}\pi$.

10. $\lim\limits_{n\to\infty} n^{-4}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n^2}\sqrt{ni+j}=$

(A) $\frac{4}{15}(2^{5/2}-1)$; (B) $\frac{8}{15}(2^{5/2}-1)$; (C) $\frac{4}{15}(2^{3/2}-1)$; (D) $\frac{8}{15}(2^{3/2}-1)$.

⊙ **Part 2: 多選擇題**
(Multiple-Choice Questions with More Than One Correct Answers)
(多選五題，每題五分，共二十五分。
各題之選項獨立判定；所有選項都答對，得五分；答錯一個選項，得三分；所有選項均未作答或答錯多於一個選項，得零分。)
(5 questions, each question is worth 5 points, for 25 points in total. The options for each question are judged independently; if all options are answered correctly, five points will be awarded; if one option is answered incorrectly, three points will be awarded; if no option is answered or more than one option is answered incorrectly, no point will be awarded.)
Example:
Answer: ABC
Student: ABC => 5 points, AB => 3 points, AD => 0 points, ABD => 0 points

11. Let f be a continuous function, $g(x,y)=f(x^2+y^2)$ and $D=\{(x,y)|x^2+y^2\le 1\}$. Which of the following statement must be true.

(A) $\displaystyle\iint_{[0,1]\times[1,2]} g(x,y)\ dA=\iint_{[1,2]\times[0,1]} g(x,y)\ dA$;

(B) $\displaystyle\iint_{[0,1]\times[0,1]} g(x^2,y)\ dA=\iint_{[0,1]\times[0,1]} g(x,y^2)\ dA$;

(C) $\displaystyle\iint_D g(x,y)\ dA=\int_0^{2\pi}\int_0^1 f(r^2)\ dr\ d\theta$;

(D) $\displaystyle\iint_D g(x,y)\ dA=\pi[F(1)-F(0)]$, where F is an antiderivative of f on $[0,1]$.

12. Which of the following limits exist?

(A) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}}$;

(B) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^4+y^2}{x^2+y^6}$;

(C) $\displaystyle\lim_{(x,y,z)\to(0,0,0)}\frac{x^2y}{x^2+y^4+z^4}$;

(D) $\displaystyle\lim_{(x,y)\to(0,0)}(x^2+y^2)\ln(x^2+y^2)$.

13. Let

$$f(x, y) = x^2 + 6(y^2 + y + 1)^2 \quad \text{and} \quad g(x, y) = x^2 + (y^3 - 1)^2.$$

Using the method of Lagrange multipliers to find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = 1$. Which of the following statements must be true?

(A) $f(1, 1)$ is an absolute extreme value;
(B) $\nabla f(x, y) = \nabla g(x, y)$ and $g(x, y) = 1$ has no solution;
(C) $f(0, 0)$ is the absolute minimum value;
(D) $f(0, \sqrt[3]{2})$ is the absolute maximum value.

14. Let $F(x, y) = \displaystyle\int_0^x \int_v^y \cos^4(u - 1)\, du\, dv$. Which of the following statements must be true?

(A) $F(1, 2) \leq \dfrac{3}{2}$;
(B) $F(2, 1) = 0$;
(C) $F(x, y) = F(y, x)$;
(D) Let $g(x) = F(x, x)$. Then $g'(2) = 2\cos^4 1$.

15. Assume the function $f(x) = \dfrac{1}{1 - x - x^2}$ has a power series expansion $\displaystyle\sum_{n=0}^{\infty} a_n x^n$. Which of the following statements must be true?

(A) $f(x) = \displaystyle\sum_{n=0}^{\infty} (x + x^2)^n$ for $|x + x^2| < 1$;
(B) The power series is convergent when $-\dfrac{1}{2} < x < \dfrac{1}{2}$;
(C) $a_n = \displaystyle\sum_{k=0}^{n} \binom{n-k}{k}$, where $\displaystyle\binom{m}{n} = 0$ when $n > m$;
(D) $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$.

◎ **Part 3: 填充題 (Fill-in-the-Blank Questions)**
(兩個題組，共十五分。)
(There are two question sets worth a total of 15 points.)

1. Let $f(x,y) = -x^3 + 4xy - 2y^2 + 1$ and $\mathbf{u} = \langle h, k \rangle$, where $h^2 + k^2 = 1$.

   (A) (5 pts) Suppose $\boldsymbol{D}^2_{\mathbf{u}} f(0,0) = \boldsymbol{D}_{\mathbf{u}}(\boldsymbol{D}_{\mathbf{u}} f)(0,0) = ah^2 + bhk + ck^2$. Then $a + b + c =$ ____(1)____.

   (B) (5 pts) Let $\gamma = \{(h,k) | h^2 + k^2 = 1$ and $f(0,0)$ is a local maximum in the direction of $\langle h, k \rangle\}$. Then the length of $\gamma =$ ____(2)____.

2. (5 pts) Let $0 < q < 1$. Evaluate $\displaystyle\sum_{n=0}^{\infty} n(1-q)q^n =$ ____(3)____

◎ **Part 4: 計算/證明題 (Calculation/Proof Questions)**
(一個題組，共十分。) (There is one question set worth a total of 10 points.)
(盡可能完整回答計算/證明題。若有輕微錯誤，則酌給部分分數。但是，如果答案正確，但論述不正確，則不給分。)
(Answer the calculation/proof questions as thoroughly as possibly. If there are minor errors, then partial marks will be awarded. However, if the answer is correct, but the argument is incorrect, then no marks will be awarded.

1. Let D be the region on the xy-plane enclosed by the curve

$$S : (x^2 + y^2 - x)^2 = x^2 + y^2.$$

   (A) (2 pts) Show that $r = \cos\theta - 1$ is a polar equation for S.

   (B) (4 pts) Find the area of D.

   (C) (4 pts) Find the average value of $f(x,y) = \sqrt{x^2 + y^2}$ over D.