Ch11-10

單選題

■ Assume that $f(x)=\int_2^x \frac{e^t-1}{t}dt$. Then $f^{(6)}(0)=?$

(A) $\frac{1}{6!}$, (B) 1,

(C) $\frac{1}{6\cdot(6!)}$, (D) $\frac{1}{6}$.

Ans：D

Sol：

$$\because e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}\Rightarrow\frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$$

$$f(x)=\int_2^x\frac{e^t-1}{t}dt=\int_2^x\sum_{n=1}^{\infty}\frac{t^{n-1}}{n!}dt=\sum_{n=1}^{\infty}\int_2^x\frac{t^{n-1}}{n!}dt=\sum_{n=1}^{\infty}\frac{\left(x^n-2^n\right)}{n\bullet n!}$$

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}\Rightarrow f''(x)=\sum_{n=2}^{\infty}\frac{(n-1)x^{n-2}}{n!}\Rightarrow\ldots\Rightarrow f^{(6)}(x)=\sum_{n=6}^{\infty}\frac{(n-1)\cdots(n-5)x^{n-6}}{n!}$$

$$f^{(6)}(0)=\frac{1}{6}$$

Note: 第二行 $\sum$ 和積分互換式 需考慮到數列的收斂半徑

■ Find the coefficient of x^5 in the Maclaurin series for $f(x)=\int \cos(x^2)dx$.

(1) $-\frac{1}{10}$ (2) $\frac{1}{15}$ (3) $-\frac{1}{5}$ (4) $\frac{2}{5}$

(5) $-\frac{2}{5}$ (6) $-\frac{1}{15}$ (7) $\frac{1}{5}$ (8) $\frac{1}{10}$

Ans：1

Sol：

$$\cos(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\Rightarrow \cos(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}=1-\frac{x^4}{2!}+\frac{x^8}{4!}-\cdots$$

$$f(x)=\int\cos(x^2)dx=\int(1-\frac{x^4}{2!}+\frac{x^8}{4!}-\cdots)dx=C+x-\frac{x^5}{5\cdot 2!}+\frac{x^9}{9\cdot 4!}-\cdots+(-1)^n\frac{x^{4n+1}}{(4n+1)(2n)!}+\cdots$$

so the coefficient of x^5 in the Maclaurin series for $f(x)$ is $-\frac{1}{10}$.

■ Which of the following is the nearest to $\int_0^{0.1}\frac{\ln(1+t)}{t}dt$?

(a) $\frac{0.1}{1}-\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}-\frac{(0.1)^4}{4}$.

(b) $\frac{0.1}{1}+\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}+\frac{(0.1)^4}{4}$.

(c) $\frac{0.1}{1^2}-\frac{(0.1)^2}{2^2}+\frac{(0.1)^3}{3^2}-\frac{(0.1)^4}{4^2}$.

(d) $\frac{0.1}{1^2}+\frac{(0.1)^2}{2^2}+\frac{(0.1)^3}{3^2}+\frac{(0.1)^4}{4^2}$.

Ans：c

Sol：

$$\ln(1+t)=\int\frac{1}{1+t}dt=\int\frac{1}{1-(-t)}dt=\int 1+(-t)+(-t)^2+\cdots dt=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots+C$$

for $|t|<1$, $C=\ln(1)=0$ when $t=0$

So $\frac{\ln(1+t)}{t}=1-\frac{t}{2}+\frac{t^2}{3}-\frac{t^3}{4}+\cdots$

$$\int_0^{0.1}\frac{\ln(1+t)}{t}dt=\int_0^{0.1}1-\frac{t}{2}+\frac{t^2}{3}-\frac{t^3}{4}+\cdots dt=\left[t-\frac{t^2}{2^2}+\frac{t^3}{3^2}-\frac{t^4}{4^2}+\cdots\right]_{t=0}^{t=0.1}$$

$$\approx 0.1-\frac{(0.1)^2}{2^2}+\frac{(0.1)^3}{3^2}-\frac{(0.1)^4}{4^2}$$

■ Suppose $g(x)=\cos(x^2)$. Find $g^{(8)}(0)$. (Hint: $\cos x=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$)

A) 360 B) 2480 C) 1680 D) 16240.

Ans：C

Sol：

$$\cos(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\Rightarrow\cos(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!}=1-\frac{x^4}{2!}+\frac{x^8}{4!}-\cdots$$

$$\because C_n=\frac{g^{(n)}(0)}{n!}\quad\therefore g^{(8)}(0)=C_8\cdot 8!=\frac{(-1)^2}{(2\cdot 2)!}\cdot 8!=1680$$

■ The first three nonzero terms in the Maclaurin series for the function $f(x)=\cos^{-1}(x)$ is (Hint: $(\cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}}$)

A) $1-\frac{1}{2}x^2+\frac{1}{24}x^4$ B) $\frac{\pi}{2}-x-\frac{1}{6}x^3$

C) $x-\frac{1}{6}x^3+\frac{1}{120}x^5$ D) $\frac{\pi}{2}-x+\frac{1}{6}x^3$.

Ans：B

Sol：

$$(\cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}} = -(1-x^2)^{-\frac{1}{2}} = -\sum_{n=0}^{\infty}\binom{-1/2}{n}(-x^2)^n = -\left(1+\frac{1}{2}x^2+\cdots\right)$$

$$\cos^{-1}(x) = \int -1-\frac{1}{2}x^2-\cdots = C - x - \frac{x^3}{6} - \cdots$$

$$C = \cos^{-1}(0) = \frac{\pi}{2} \quad \text{when} \quad x=0$$

Hence $f(x) = \frac{\pi}{2} - x - \frac{x^3}{6} - \cdots$

■ If $f(x)=\cos(x^2)$, then $f^{(12)}(0)=$

A) $\frac{1}{12!}$ B) $-\frac{1}{6!}$ C) $\frac{12!}{6!}$ D) $-\frac{12!}{6!}$.

Ans：D

Sol：

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!} = 1-\frac{x^4}{2!}+\frac{x^8}{4!}-\cdots$$

$$\because C_n = \frac{g^{(n)}(0)}{n!} \quad \therefore g^{(12)}(0) = C_{12}\cdot 12! = \frac{(-1)^3}{(2\cdot 3)!}\cdot 12! = -\frac{12!}{6!}$$

■ Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$.

(A) $\frac{1}{2}$; (B) $\frac{\sqrt{3}}{2}$; (C) $\frac{\sqrt{5}}{2}$; (D) 3.

Ans：B

Sol:

$$f(x) = \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{6}\right)^{2n} = f(\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$$

填空題

■ Find the Taylor polynomial at 0 of degree 4 for $f(x) = \ln(1+x)$. ______

Ans： $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

Sol：

$$\ln(1+x) = \int \frac{1}{1+x} dx = \int \frac{1}{1-(-x)} dx = \int 1 + (-x) + (-x)^2 + \cdots dx = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for $|x| < 1$, $C = \ln(1) = 0$ when $x = 0$

So $\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

■ If the Maclaurin series for $f(x)$ is $1 - 9x + 16x^2 - 25x^3 + \ldots$, find $f^3(0)$.

Ans：**-150**

Sol：

The Maclaurin series $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^3(0)}{3!}x^3 + \cdots$

So $-25 = \frac{f^3(0)}{3!} \Rightarrow f^3(0) = -150$

■ Find the first three terms of the Taylor series of $f(x)=\sqrt{x}$ centered at 4: -

Ans： $2+\frac{x-4}{4}-\frac{(x-4)^2}{64}$

Sol：

The first three terms of $f(x)$ centered at 4 is $f(4)+\frac{f'(4)}{1!}(x-4)+\frac{f''(4)}{2!}(x-4)^2$,

where $f'(x)=\frac{1}{2\sqrt{x}}$, $f''(x)=-\frac{1}{4x^{3/2}}$.

Hence $f(4)+\frac{f'(4)}{1!}(x-4)+\frac{f''(4)}{2!}(x-4)^2=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2$

■ Find the Maclaurin series for the function $f(x)=x\sin(x^2)$.

Ans： $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n-1}}{(2n-1)!}$

Sol：

$$\sin(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\Rightarrow\sin(x^2)=\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}$$

$$f(x)=x\sin(x^2)=x\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{4(n-1)+3}}{(2(n-1)+1)!}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{4n-1}}{(2n-1)!}\ \text{or}\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n-1}}{(2n-1)!}$$ 皆可

■ Let $f(x)=\sqrt{1+x^3}$. Use the binomial series to evaluate $f^{(9)}(0)=$ _____ .

Ans： $9!\dfrac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\times 2\times 3}$. **or** $\boxed{\dfrac{9!}{16}}$ or $\boxed{22680}$

Sol：

$$f(x)=\left(1+x^3\right)^{1/2}=\sum_{n=0}^{\infty}\binom{1/2}{n}x^{3n}$$

$\dfrac{f^{(9)}(0)}{9!}$ is the coefficient of x^9, so

$$\frac{f^{(9)}(0)}{9!}=\binom{1/2}{3}\Rightarrow f^{(9)}(0)=9!\cdot\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}=\frac{9!}{16}=22680$$

■ Find the Maclaurin series of $\dfrac{2}{(1-x)^3}=\sum_{n=0}^{\infty}a_n x^n$. $a_n=$ __________.

Ans： $(n+2)(n+1)$

Sol：

$$\frac{2}{(1-x)^3}=2(1-x)^{-3}=2\sum_{n=0}^{\infty}\binom{-3}{n}(-x)^n=2\left(1+3x+\frac{3\cdot 4}{2!}x^2+\frac{3\cdot 4\cdot 5}{3!}x^3+\cdots\right)$$

$$=2\sum_{n=0}^{\infty}\binom{3+n-1}{n}x^n=2\sum_{n=0}^{\infty}\frac{(2+n)!}{n!\cdot 2!}x^n=\sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

Hence $a_n=(n+2)(n+1)$.

■ If

$f(x) = 3x^4 - 17x^3 + 35x^2 - 32x + 17 = a_4(x-1)^4 + a_3(x-1)^3 + a_2(x-1)^2 +$

,

then $a_3 =$ ________.

Ans：-5

Sol：

$$f(x) = a_4(x-1)^4 + a_3(x-1)^3 + a_2(x-1)^2 + a_1(x-1) + a_0 = \sum_{n=0}^{4} \frac{f^n(1)}{n!}(x-1)^n$$

So $a_3 = \dfrac{f^3(1)}{3!}$

$$f'(x) = 12x^3 - 51x^2 + 70x - 32$$
$$f''(x) = 36x^2 - 102x + 70$$
$$f^3(x) = 72x - 102$$

$$\Rightarrow a_3 = \frac{f^3(1)}{3!} = \frac{72-102}{6} = -5$$

■ If $\sum_{n=0}^{\infty} a_n x^n$ is the Maclaurin series of $1/(1+x)^2$, then $a_n =$ ________.

Ans：$(-1)^n(n+1)$

Sol：

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n = 1 - 2x + \frac{2\cdot 3}{2!}x^2 - \frac{2\cdot 3\cdot 4}{3!}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{2+n-1}{n} x^n = \sum_{n=0}^{\infty} (-1)^n \frac{(1+n)!}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

Hence $a_n = (-1)^n (n+1)$.

■ Find the first four terms ________ of the Maclaurin series of

$$f(x)=\frac{\cos x}{1-x}.$$

Ans： $1+x+x^2/2+x^3/2$

Sol：

$$\cos(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots$$

$$\Rightarrow f(x)=\frac{\cos(x)}{1-x}=\left(1-\frac{x^2}{2!}+\cdots\right)\cdot\left(1+x+x^2+x^3+\cdots\right)$$

$$=1+x+x^2-\frac{x^2}{2}+x^3-\frac{x^3}{2}+\cdots=1+x+\frac{x^2}{2}+\frac{x^3}{2}+\cdots$$

■ Let $f(x)=\frac{x}{\sqrt{1+x^4}}$ and $f(x)=\sum_{n=0}^{\infty}a_nx^n$ be its Maclaurin series. Then $(a_1,a_2,\cdots,a_6)=$ ____________ .

Ans： $(1,0,0,0,-1/2,0)$

Sol：

$$f(x)=\frac{x}{\sqrt{1+x^4}}=x\cdot\frac{1}{\sqrt{1+x^4}}=x\cdot\left(1+x^4\right)^{-\frac{1}{2}}=x\cdot\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(x^4\right)^n$$

Note: $(a_1,a_2,\cdots,a_6)$ 分別為 $x^1,x^2,\cdots,x^6$ 項係數

$$=x\cdot\left(1+\left(-\frac{1}{2}\right)x^4+\cdots\right)=x+0\cdot x^2+0\cdot x^3+0\cdot x^4+\left(-\frac{1}{2}\right)x^5+0\cdot x^6+\cdots$$

Hence $(a_1,a_2,\cdots,a_6)=(1,0,0,0,-1/2,0)$

■ Let $T(k)=4\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$. Using the binomial expansion on the integrand, it can be shown that $T\left(k\right)=\sum_{n=0}^{\infty}a_n k^{2n}$. Find the *value* of a_1 = ________.

Ans： $\frac{\pi}{2}$

Sol：

$$T(k)=4\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-k^2\sin^2 x}}=4\int_0^{\frac{\pi}{2}}\left(1-k^2\sin^2 x\right)^{-\frac{1}{2}}\ dx$$

$$=4\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-k^2\sin^2 x\right)^n\ dx$$

因為題目要求的是a_1，也就是說找出k^2項係數即可

$$=4\int_0^{\frac{\pi}{2}}1+\left(-\frac{1}{2}\right)\left(-k^2\sin^2 x\right)+\cdots dx=4\left(C+x+\frac{k^2}{2}\int_0^{\frac{\pi}{2}}\sin^2 x\,dx+\cdots\right)$$

where $\int_0^{\frac{\pi}{2}}\sin^2 x\,dx=\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}\,dx=\left[\frac{1}{2}x-\frac{\sin 2x}{4}\right]_{x=0}^{x=\frac{\pi}{2}}=\frac{\pi}{4}$

$$T(k)=4\left(C+x+\frac{k^2}{2}\int_0^{\frac{\pi}{2}}\sin^2 x\,dx+\cdots\right)=4C+4x+4\cdot\frac{k^2}{2}\cdot\frac{\pi}{4}+\cdots$$

Hence $a_1=\frac{\pi}{2}$

■ The Maclaulin series of $\int_0^x\frac{\arctan t}{t}dt=$ ____________.

Ans： $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)^2}$

Sol：

$$f'(x)=\frac{d}{dx}\int_0^x\arctan t\,dt=\arctan x\text{（微積分基本定理）}$$

$$\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\Rightarrow\frac{\arctan x}{x}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n+1}=1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\cdots$$

$$f(x)=\int\frac{\arctan x}{x}=\int\left(1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+\cdots\right)=x-\frac{x^3}{3^2}+\frac{x^5}{5^2}-\frac{x^7}{7^2}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{\left(2n+1\right)^2}$$