Ch11-3

單選題

■ Which of the three series below converges?

(i) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ (ii) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ (iii) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$

(A) (i) (B) (i), (ii), (iii) (C) (ii), (iii) (D) none.

Ans：C

SOL：

(i) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

Let $f(n)=\frac{1}{n\ln n}$, consider $\int_2^{\infty} f(x)dx=\int_2^{\infty}\frac{1}{x\ln x}dx=\lim_{a\to\infty}\int_2^a\frac{1}{x\ln x}dx$

$$=\lim_{a\to\infty}\int_{\ln 2}^{\ln a}\frac{1}{u}du\begin{pmatrix}u=\ln x\\ du=\frac{1}{x}dx\end{pmatrix}=\lim_{a\to\infty}\ln u\,\Big|_{\ln 2}^{\ln a}=\lim_{a\to\infty}\ln(\ln a)-\ln(\ln 2)=\infty$$

By Integral test $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverge.

(ii) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

Let $f(n)=\frac{1}{n(\ln n)^2}$, consider $\int_2^{\infty} f(x)dx=\int_2^{\infty}\frac{1}{x(\ln x)^2}dx=\lim_{a\to\infty}\int_2^a\frac{1}{x(\ln x)^2}dx$

$$=\lim_{a\to\infty}\int_{\ln 2}^{\ln a}\frac{1}{u^2}du\begin{pmatrix}u=\ln x\\ du=\frac{1}{x}dx\end{pmatrix}=\lim_{a\to\infty}\frac{-1}{u}\Big|_{\ln 2}^{\ln a}=\lim_{a\to\infty}\frac{-1}{\ln a}-\frac{-1}{\ln 2}=0-\frac{-1}{\ln 2}$$

By Integral test $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converge.

(iii) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$

Let $f(n)=\frac{1}{n(\ln n)^3}$, consider $\int_2^{\infty} f(x)dx=\int_2^{\infty}\frac{1}{x(\ln x)^3}dx=\lim_{a\to\infty}\int_2^a\frac{1}{x(\ln x)^3}dx$

$$=\lim_{a\to\infty}\int_{\ln 2}^{\ln a}\frac{1}{u^3}du\begin{pmatrix}u=\ln x\\ du=\frac{1}{x}dx\end{pmatrix}=\lim_{a\to\infty}\frac{-1/2}{u^2}\Big|_{\ln 2}^{\ln a}=\lim_{a\to\infty}\frac{-1}{2(\ln a)^2}-\frac{-1}{2(\ln 2)^2}=0-\frac{-1}{2(\ln 2)^2}$$

By Integral test $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$ converge.

Note: 因為 (iii)< (ii)<(i) 由(ii)收斂也可得(iii)為收斂 (遞增有上界)

填空題

■ Find all *possible values* of b such that $\sum_{n=1}^{\infty}(b^2)^{\ln n}$ is convergent . ____(4)____

Ans：$|b|< e^{-1/2}$ or $|b|<\frac{1}{\sqrt{e}}$ or $(-\frac{1}{\sqrt{e}},\frac{1}{\sqrt{e}})$ or $(-e^{-\frac{1}{2}},e^{-\frac{1}{2}})$

SOL：

$$(b^2)^{\ln n} = \left(e^{2\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{2\ln b} = n^{2\ln b} = \frac{1}{n^{-2\ln b}}$$

This is a *p*-series, which converges for all b such that $-2\ln b > 1$

$$\Rightarrow \ln b < -\frac{1}{2} \Rightarrow b < -e^{\frac{1}{2}} \text{ [with } b > 0\text{]}$$