Ch11-4

單選題

■ Which one of the following series is convergent?

A) $\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{1+n^2}\right)$ B) $\sum_{n=1}^{\infty}\frac{n+1}{\sqrt[3]{2n^6-n^2+1}}$

C) $\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{\pi}{2n}\right)$ D) $\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n}\right)$.

Ans：A

SOL：

(A) Let $a_n=\tan^{-1}\left(\frac{1}{1+n^2}\right)$ and $b_n=\frac{1}{1+n^2}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\tan^{-1}\left(\frac{1}{1+n^2}\right)\Big/\left(\frac{1}{1+n^2}\right)=\lim_{x\to 0}\frac{\tan^{-1}x}{x}\overset{L'H}{=}\lim_{x\to 0}\frac{1}{x^2+1}=1>0.$$

$\frac{1}{1+n^2}<\frac{1}{n^2}$， $\sum n^{-2}$ 是 $\sum\frac{1}{n^p}$ 型， $p=2>1\Rightarrow\sum_{n=1}^{\infty}n^{-2}$ converges.

Since $\sum b_n$ is convergent, $\sum a_n$ also converges by Limit Comparison Test.

(11-4)

(B) $\frac{1}{\sqrt[3]{2}}\cdot\frac{1}{n}=\frac{n}{\sqrt[3]{2n^6}}<\frac{n+1}{\sqrt[3]{2n^6-n^2+1}}$ for $n\ge 1$. since $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{2}}\cdot\frac{1}{n}=\frac{1}{\sqrt[3]{2}}\cdot\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series, $\sum_{n=1}^{\infty}\frac{n+1}{\sqrt[3]{2n^6-n^2+1}}$ diverges by the Comparison Test.

(11-4)

(C) $\lim_{n\to\infty}\cos\left(\frac{\pi}{2n}\right)=\cos\left(\lim_{n\to\infty}\left(\frac{\pi}{2n}\right)\right)=\cos(0)=1$

Since $\lim_{n\to\infty}(-1)^n\cos\left(\frac{\pi}{2n}\right)\neq 0$, the series diverges by the Test of Divergence. (11-2)

(D) $\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n}\right)=\sum_{n=2}^{\infty}\ln\left(\frac{n-1}{n}\right)=\lim_{a\to\infty}\sum_{n=2}^{a}(\ln(n-1)-\ln n)=\lim_{a\to\infty}(\ln(1)-\ln a)=-\infty$.

(11-2)

■ Which of the following three tests will establish that the series $\sum_{n=1}^{\infty}\frac{n}{\sqrt{2n^5+1}}$ converges?

(i) Comparison test with $\sum_{n=1}^{\infty} n^{-\frac{5}{2}}$ (ii) Comparison test with $\sum_{n=1}^{\infty} n^{-\frac{3}{2}}$

(iii) Comparison test with $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$

(a) (i), (ii) (b) (i), (iii) (c) (ii) (d) (i), (ii), (iii)

Ans：c

SOL：

(i) $\sum_{n=1}^{\infty} n^{-\frac{5}{2}}$ 是 $\sum_{n=1}^{\infty}\frac{1}{n^p}$ 型， $p=\frac{5}{2}>1 \Rightarrow \sum_{n=1}^{\infty} n^{-\frac{5}{2}}$ converges.

$\because n^{-\frac{5}{2}} \le \frac{1}{2} n^{-\frac{3}{2}} = \frac{n}{\sqrt{2n^5+2n^5}} < \frac{n}{\sqrt{2n^5+1}}$ when $n \ge 2$,

$\therefore \sum_{n=1}^{\infty} n^{-\frac{5}{2}}$ converges 無法推得任何結果

(ii) $\sum_{n=1}^{\infty} n^{-\frac{3}{2}}$ 是 $\sum_{n=1}^{\infty}\frac{1}{n^p}$ 型， $p=\frac{3}{2}>1 \Rightarrow \sum_{n=1}^{\infty} n^{-\frac{3}{2}}$ converges.

$\because \frac{n}{\sqrt{2n^5+1}} < \frac{n}{n^{5/2}} = n^{-\frac{3}{2}}$, $\therefore \sum_{n=1}^{\infty} n^{-\frac{3}{2}}$ converges $\Rightarrow \sum_{n=1}^{\infty}\frac{n}{\sqrt{2n^5+1}}$ converges.

(iii) $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$ 是 $\sum_{n=1}^{\infty}\frac{1}{n^p}$ 型， $p=\frac{1}{2}<1 \Rightarrow \sum_{n=1}^{\infty} n^{-\frac{1}{2}}$ diverges.

$\because \frac{n}{\sqrt{2n^5+1}} < n^{-\frac{3}{2}} < n^{-\frac{1}{2}}$, $\therefore \sum_{n=1}^{\infty} n^{-\frac{1}{2}}$ diverges 無法推得任何結果