**Ch11-5**

**單選題**

■ Which of the following series converges?

(1) a,c,d (2) a,b,d (3) b,c,d (4) a,b,c

| | | | |
|---|---|---|---|
| (a) | $\sum \frac{n}{n+1}$ | (b) | $\sum \frac{\sqrt{n}+1}{n^2+2}$ |
| (c) | $\sum (-1)^{n-1}\frac{\ln n}{n}$ | (d) | $\sum \frac{(-1)^{n-1}}{\sqrt{n+1}}$ |

Ans：3

SOL：

(a) $\because \lim_{n\to\infty}\frac{n}{n+1}=1 \quad \therefore \sum\frac{n}{n+1}$ diverges. (11-2)

(b) $\frac{\sqrt{n}+1}{n^2+2}<\frac{2\sqrt{n}}{n^2}=2n^{-\frac{3}{2}}$, $\sum n^{-\frac{3}{2}}$ 是 $\sum\frac{1}{n^p}$ 型，$p=\frac{3}{2}>1\Rightarrow\sum_{n=1}^{\infty}n^{-\frac{3}{2}}$ converges.

So $\sum\frac{\sqrt{n}+1}{n^2+2}$ converges by the Comparison Test. (11-4)

(c) Let $f(x)=\frac{\ln x}{x}>0$ for $x>0$ and $f'(x)=\frac{1-\ln x}{x^2}<0$ for $x>e$

so $f$ is decreasing on $(e,\infty)$. Also, $\lim_{x\to\infty}\frac{\ln x}{x}\overset{L'H}{=}\lim_{x\to\infty}\frac{1/x}{1}=0$

So the series converges by the Alternating Series Test. (11-5)

(d) Let $f(x)=\frac{1}{\sqrt{x+1}}>0$ for $x>0$ and $f'(x)=-\frac{1}{2(x+1)^{-3/2}}<0$ for $x>0$

so $f$ is decreasing on $(0,\infty)$. Also, $\lim_{x\to\infty}\frac{1}{\sqrt{x+1}}=0$

So the series converges by the Alternating Series Test. (11-5)

■ Determine which series is divergent .

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$ . (b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{10n^2-1}$. (c) $\sum_{n=1}^{\infty} \frac{n}{n^2+7n+1}$. (d) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2+1}$.

Ans：c

SOL：

(a) $f(x)=xe^{-x^2}$ is continuous and positive on $(0,\infty)$ .

$f'(x)=e^{-x^2}-2x^2e^{-x^2}=e^{-x^2}\left(1-2x^2\right)<0$ for $x>0$, so $f$ is decreasing on $(0,\infty)$ . Thus, the Integral Test applies

$$\int_1^{\infty} xe^{-x^2}dx=\lim_{a\to\infty}\int_1^a xe^{-x^2}dx$$

$$=\lim_{a\to\infty}\int_{-1}^{-a^2} -\frac{1}{2}e^u du\begin{pmatrix} u=-x^2 \\ du=-2xdx \end{pmatrix}=-\frac{1}{2}\lim_{a\to\infty}\left[e^u\right]_{-1}^{-a^2}=\lim_{a\to\infty}-\frac{e^{-a^2}-e^{-1}}{2}=\frac{1}{2e}.$$ (11-3)

(b) Let $f(x)=\frac{x}{10x^2-1}>0$ for $x>0$ and $f'(x)=-\frac{10x^2+1}{(10x^2-1)^2}<0$ for $x>0$

so $f$ is decreasing on $(0,\infty)$ . Also, $\lim_{x\to\infty}\frac{x}{10x^2-1}=0$

So the series converges by the Alternating Series Test. (11-5)

(c) Let $a_n=\frac{n}{n^2+7n+1}$ and $b_n=\frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n/n^2+7n+1}{1/n}=\lim_{n\to\infty}\frac{n^2}{n^2+7n+1}=1>0$

Since $\sum b_n$ is the divergent harmonic series, $\sum a_n$ also diverges by Limit Comparison Test. (11-4)

(d) $\frac{\sin n}{n^2+1}<\frac{1}{n^2+1}<\frac{1}{n^2}=n^{-2}$, $\sum n^{-2}$ 是 $\sum\frac{1}{n^p}$ 型， $p=2>1\Rightarrow\sum_{n=1}^{\infty}n^{-2}$ converges

So $\sum_{n=1}^{\infty}\frac{\sin n}{n^2+1}$ converges by the Comparison Test. (11-4)

■ Which of the following series converges ?

(a) $\sum_{n=1}^{\infty}\sin(\frac{1}{n})$ , (b) $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ , (c) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+1}$ , (d) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ .

Ans：d

SOL：

(a) Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$.

Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{\theta\to\infty}\frac{\sin(\theta)}{\theta} = 1 > 0$$

Since $\sum b_n$ is the divergent harmonic series, $\sum a_n$ also diverges. (11-4)

(b) Let $f(n) = \frac{\ln n}{n}$, consider $\int_1^{\infty} f(x)dx = \int_1^{\infty}\frac{\ln x}{x}dx = \lim_{a\to\infty}\int_1^{a}\frac{\ln x}{x}dx$

$$= \lim_{a\to\infty}\int_{\ln 1}^{\ln a} u\,du \begin{pmatrix} u = \ln x \\ du = \frac{1}{x}dx \end{pmatrix} = \lim_{a\to\infty}\frac{u^2}{2}\Big|_0^{\ln a} = \lim_{a\to\infty}\frac{(\ln a)^2}{2} = \infty$$

By Integral test $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ diverge. (11-3)

(c) Let $b_n = \frac{n}{n+1}$, $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1} = \sum_{n=1}^{\infty}(-1)^n b_n$ , now $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$ .

Since $\lim_{n\to\infty}(-1)^n\frac{n}{n+1} \neq 0$, the series diverges by the Test of Divergence. (11-5)

(d) Let $f(n) = \frac{1}{n(\ln n)^2}$, consider $\int_2^{\infty} f(x)dx = \int_2^{\infty}\frac{1}{x(\ln x)^2}dx = \lim_{a\to\infty}\int_2^{a}\frac{1}{x(\ln x)^2}dx$

$$= \lim_{a\to\infty}\int_{\ln 2}^{\ln a}\frac{1}{u^2}du \begin{pmatrix} u = \ln x \\ du = \frac{1}{x}dx \end{pmatrix} = \lim_{a\to\infty}\frac{-1}{u}\Big|_{\ln 2}^{\ln a} = \lim_{a\to\infty}\frac{-1}{\ln a} - \frac{-1}{\ln 2} = 0 - \frac{-1}{\ln 2}$$

By Integral test $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converge. (11-3)

■ The set of $p$ for which the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^p}$ converges is

(A) $(-\infty, \infty)$, (B) $(-\infty, 1)$,

(C) $(1, \infty)$, (D) $[1, \infty)$.

Ans：A

SOL：

Let $f(x) = \frac{1}{x(\ln x)^p} > 0$ for $x > 1$ and $f'(x) = \frac{-\left((\ln x)^{p-1} \cdot (\ln x + p)\right)}{(x(\ln x)^p)^2} < 0$ if $x > e^{-p}$, so $f$ is eventually decreasing for every fixed $p$.

Clearly $\lim_{x \to \infty} \frac{1}{x(\ln x)^p} = 0$, so the series converges for all $p$.

By the Alternating Series Test. (11-5)

**多選題**

■ Which of the following statements about the infinite sequence $\{a_n\}_{n=1}^{\infty}$ are true ?

(A) If $\lim_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} |a_n| = 0$;

(B) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \left(|a_n| + a_n\right)$ converges;

(C) If $\sum_{n=1}^{\infty} \left(|a_n| + a_n\right)$ converges, then $\sum_{n=1}^{\infty} a_n$ converges;

(D) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Ans：A

SOL：

(A) Ture!

<pf.1>

Since $|a_n| = (a_n + |a_n|) - a_n$ ,and $a_n + |a_n| = \begin{cases} 2a_n \text{ ,if } a_n > 0 \\ 0 \text{ , if } a_n < 0 \end{cases}$

$\therefore \lim_{n\to\infty}|a_n| = \lim_{n\to\infty}(a_n + |a_n|) - a_n = 0 - 0 = 0$ ■ (11-1)

<pf.2> (By definition.)

$\lim_{n\to\infty} a_n = 0$ => $\forall\varepsilon > 0, \exists N > 0$ such that $|a_n - 0| < \varepsilon$, for all n>N

$\therefore \left||a_n| - 0\right| = |a_n - 0| < \varepsilon$ for all n>N. ■ (11-1)

(B) False!

Consider $a_n = \frac{(-1)^n}{n}$ => $\sum_{n=1}^{\infty} a_n$ converge (By alternative series test)

$\therefore |a_n| + a_n = \begin{cases} \frac{2}{n}, \text{if n is even.} \\ 0, \text{ if n is odd.} \end{cases}$

=>$\sum_{n=1}^{\infty}(|a_n| + a_n) = \sum_{n=(even)}^{\infty} \frac{2}{n} = \text{diverges}$. (Since $\sum \frac{1}{n}$ diverges.) (11-5)

(C) False!

Consider $a_n = \frac{-1}{n}$ => $|a_n| + a_n = 0$ =>$\sum_{n=1}^{\infty}(|a_n| + a_n) = 0$ (converges)

But, $\sum_{n=1}^{\infty} a_n$ diverges. (11-2)

(D) False!

Consider $a_n = \frac{(-1)^n}{n}$ => $\sum_{n=1}^{\infty} a_n$ converges. (By alternative series test)

But $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (11-5)

■ Which of the following statements is true ?

(A) Assume that the series $\sum_{n=1}^{\infty} a_n$ converges. Then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ also converges.

(B) Assume that the series $\sum_{n=1}^{\infty} a_n$ converges. Then the series $\sum_{n=1}^{\infty} a_n^2$ also converges.

(C) Assume that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ also converges.

(D) Assume that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge. Then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ also diverges.

Ans：C

SOL：

(A)False!

Consider $a_n = \frac{(-1)^n}{n}$ => $\sum_{n=1}^{\infty} a_n$ converges. (By alternative series test.)

But, $\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (11-5)

(B)False!

Consider $a_n = \frac{(-1)^n}{\sqrt{n}}$ => $\sum_{n=1}^{\infty} a_n$ converges. (By alternative series test.)

But, $\sum_{n=1}^{\infty} a_n{}^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (11-5)

(C)Ture!

<pf>

Since $\sum_{n=1}^{\infty} a_n , \sum_{n=1}^{\infty} a_n$ converges

=> $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \lim_{k\to\infty} \sum_{k=1}^{n} b_k$

$= \lim_{k\to\infty} \sum_{k=1}^{n} (a_k + b_k) = \sum_{n=1}^{\infty} (a_n + b_n)$

converges.

(11-2)

(D)False!

Consider $a_n = -1,\ b_n = 1$ => $\sum_{n=1}^{\infty} a_n + b_n = 0$ converges. (11-2)

■ Which of the following series are convergent ?

(A) $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{n^2}\right)$,

(B) $\sum_{n=3}^{\infty} (-1)^n \tan\frac{\pi}{n}$,

(C) $\sum_{n=1}^{\infty} \frac{n}{100n^2+1}$,

(D) $\sum_{n=1}^{\infty} \frac{\sin n}{n!}$.

Ans：ABD

SOL：

(A). Let $a_n = \ln\left(\frac{n^2+1}{n^2}\right)$, $\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$

By the limit comparison test, $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{n^2}\right)$ converges $\Leftrightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (11-4)

(B). Let $a_n = \tan\frac{\pi}{n}$. $a_n > 0$ satisfies (i) $a_{n+1} < a_n$ and (ii) $\lim_{n\to\infty} \tan\frac{\pi}{n} = 0$.

By the Alternating series test, $\sum_{n=3}^{\infty} (-1)^n \tan\frac{\pi}{n}$ converges. (11-5)

(C). Let $a_n = \frac{n}{100n^2+1}$ with positive terms, $\lim_{n\to\infty} \frac{\frac{n}{100n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{100n^2+1} = \frac{1}{100} > 0$.

By the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Leftrightarrow$ $\sum_{n=1}^{\infty} \frac{n}{100n^2+1}$ diverges. (11-4)

(D). Let $a_n = \frac{\sin n}{n!}$. $\left|\frac{\sin n}{n!}\right| \le \frac{1}{n!}$ and $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges (by the ratio test)

By the comparison test, $\sum_{n=1}^{\infty} \frac{\sin n}{n!}$ converges. (11-4)

■ Which of the following series are convergent ?

A) $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ B) $\sum_{n=3}^{\infty}(-1)^n\tan\frac{\pi}{n}$

C) $\sum_{n=1}^{\infty}\ln(\frac{n}{3n-1})$ D) $\sum_{n=1}^{\infty}\frac{\cos n}{n!}$ .

Ans：BD

SOL：

(A). Let $a_n=\frac{n}{n^2+1}$, with positive terms, $\lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2}{n^2+1}=1>0$.

By the limit comparison test, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges $\Leftrightarrow$ $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges. (11-4)

(B). Let $a_n=\tan\frac{\pi}{n}$. $a_n>0$ satisfies (i) $a_{n+1}<a_n$ and (ii) $\lim_{n\to\infty}\tan\frac{\pi}{n}=0$.

By the Alternating series test, $\sum_{n=3}^{\infty}(-1)^n\tan\frac{\pi}{n}$ converges. (11-5)

(C). Let $a_n=\ln(\frac{n}{3n-1})$. $a_n>0$ satisfies (i) $a_{n+1}<a_n$ and (ii) $\lim_{n\to\infty}\frac{\ln n}{n}=0$.

<Pf.> (i). $\frac{d}{dx}\left(\frac{\ln x}{x}\right)=\frac{1-\ln x}{x^2}<0$ for $x$ large.

By the Alternating series test, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converges. (11-5)

(C). $a_n=\ln(\frac{n}{3n-1})=\ln(\frac{1}{3-1/n})=\ln(3-1/n)^{-1}=-\ln(3-\frac{1}{n})$

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}-\ln(3-\frac{1}{n})=-\ln 3\neq 0$

So the series diverges by the Test of Divergence. (11-5)

(D). Let $a_n=\frac{\cos n}{n!}$. $\left|\frac{\cos n}{n!}\right|\le\frac{1}{n!}$ and $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges (by the ratio test)

By the comparison test, $\sum_{n=1}^{\infty}\frac{\cos n}{n!}$ converges. (11-4)