Ch11-6

單選題

■ For which of the following series will the Ratio Test fail to give a definite answer (i.e., be inconclusive)?

1) $\sum_{n=1}^{\infty}\left(\frac{99}{100}\right)^n$ 2) $\sum_{n=1}^{\infty}\left(\frac{100}{99}\right)^n$ 3) $\sum_{n=1}^{\infty} n^{-100}$

(A) None (B) 1 (C) 2 (D) 3.

Ans：D

SOL：

(1) $\lim_{n\to\infty}\frac{(99/100)^{n+1}}{(99/100)^n}=\frac{99}{100}<1\Rightarrow$ absolute convergence

(2) $\lim_{n\to\infty}\frac{(100/99)^{n+1}}{(100/99)^n}=\frac{100}{99}>1\Rightarrow$ divergence

(3) $\lim_{n\to\infty}\frac{(n+1)^{-100}}{n^{-100}}=1\Rightarrow$ The test fails.

■ Which of the following series is conditional convergent?

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n+1}}{\sqrt{n}+1}$.

(B) $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$.

(C) $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n-\ln n}$.

(D) $\sum_{n=1}^{\infty}(-1)^n\frac{(n!)^2 3^n}{(2n+1)!}$.

Ans：C

SOL：

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n+1}}{\sqrt{n}+1}$

Since $\lim_{n\to\infty}(-1)^{n+1}\frac{\sqrt{n+1}}{\sqrt{n}+1}=\begin{cases}1\\-1\end{cases}$ (Limit not exist)

$\Rightarrow\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n+1}}{\sqrt{n}+1}$ diverge.(by test of diverge)

(B) $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$.

Let $a_n=\frac{(-2)^{n+1}}{n+5^n}$

$$|a_n|=\left|\frac{(-2)^{n+1}}{n+5^n}\right|=\frac{2^{n+1}}{n+5^n}<\frac{2^n}{5^n}=\left(\frac{2}{5}\right)^n$$

Since $\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$ converge $\Rightarrow\sum_{n=1}^{\infty}|a_n|$ converge.(by comparison test)

(C) $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n-\ln n}$

1. Let $a_n=(-1)^n\frac{\ln n}{n-\ln n}$

   $|a_n|=\left|\frac{\ln n}{n-\ln n}\right|>\frac{1}{n},\forall n>e$

   Note: $n-\ln n<n\to\frac{1}{n-\ln n}>\frac{1}{n}\to\frac{\ln n}{n-\ln n}>\frac{1}{n}$ as $\ln n>1$

   Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverge $\Rightarrow\sum_{n=1}^{\infty}|a_n|$ diverge.

   $\Rightarrow\sum_{n=1}^{\infty}a_n$ not converge absolutely.

2. Let $f(x)=\frac{\ln x}{x-\ln x}$

   $f'(x)=\frac{1-\ln x}{(x-\ln x)^2}<0,\forall x>e\ldots$①

   $\lim_{x\to\infty}\frac{\ln x}{x-\ln x}\overset{L'H}{=}\lim_{x\to\infty}\frac{1/x}{1-1/x}=0\ldots$②

By ①② $\Rightarrow\sum_{n=1}^{\infty}a_n$ converge.(by alternative series test)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ conditional convergent.

(D) $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 3^n}{(2n+1)!}$

Let $a_n = (-1)^n \frac{(n!)^2 3^n}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{[(n+1)!]^2 3^{n+1}}{(2n+3)!}}{\frac{(n!)^2 3^n}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 \cdot 3}{(2n+3)(2n+2)}\right| = \frac{3}{4} < 1$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converge absolutely.(by ratio test)

■ $\sum_{n=1}^{\infty} \frac{3^n x^n}{(n+1)^2}$. The interval of x convergence is

(A) $(-\infty,\infty)$; (B) $[-1/3, 1/3]$; (C) $[-1/3, 1/3)$; (D) $(-1/3, 1/3)$.

Ans：B

SOL：

Let $a_n = \frac{3^n x^n}{(n+1)^2}$. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{n+1} x^{n+1}}{(n+2)^2}}{\frac{3^n x^n}{(n+1)^2}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 \cdot 3x}{(n+2)^2}\right| = |3x| < 1 \left(-\frac{1}{3} < x < \frac{1}{3}\right)$$

, then by the Ratio Test, the given series is absolutely convergent and therefore convergent.

For $x = \frac{1}{3}$. Since $\frac{1}{(n+1)^2} < \frac{1}{n^2}$, $\sum n^{-2}$ 是 $\sum \frac{1}{n^p}$ 型，$p = 2 > 1 \Rightarrow \sum_{n=1}^{\infty} n^{-2}$ converges

So $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by the Comparison Test.

For $x = -\frac{1}{3}$. $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{(n+1)^2}\right| = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent. So $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}$ is absolutely convergent and therefore convergent.

所以選 B

多選題

■ Which of the following series are convergent ?

A) $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ B) $\sum_{n=3}^{\infty}(-1)^n\tan\frac{\pi}{n}$

C) $\sum_{n=1}^{\infty}\ln(\frac{n}{3n-1})$ D) $\sum_{n=1}^{\infty}\frac{\cos n}{n!}$.

Ans：BD

SOL：

(A) Let $a_n=\frac{n}{n^2+1}$, $b_n=\frac{1}{n}$ => $\lim_{n\to\infty}\frac{a_n}{b_n}=1<\infty$

By limit comparison test, $\sum_{n=1}^{\infty}b_n$ diverges.=> $\sum_{n=1}^{\infty}a_n$ diverges. (11-4)

(B) Let $a_n=(-1)^n\tan\frac{\pi}{n}$, $b_n=\tan\frac{\pi}{n}$

Since $b_n>0$, and $\lim_{n\to\infty}b_n=0$ => $\sum_{n=1}^{\infty}a_n$ converges.(By alternative series test.) (11-5)

(C) Since $\lim_{n\to\infty}\ln\left(\frac{n}{3n-1}\right) = \ln\left(\frac{1}{3}\right)\neq 0$ => $\sum_{n=1}^{\infty}\ln\left(\frac{n}{3n-1}\right)$ diverges. (11-2)

(D) Since $|\frac{\cos n}{n!}|<\frac{1}{n!}$,and $\sum_{n=1}^{\infty}\frac{1}{n!}$ converge.

=> $\sum_{n=1}^{\infty}|\frac{\cos n}{n!}|$ converges. (By comparison test.)

=> $\sum_{n=1}^{\infty}\frac{\cos n}{n!}$ converges.(Since $\sum_{n=1}^{\infty}a_n$ Absoltely converge => $\sum_{n=1}^{\infty}a_n$ Converges)

(11-4 & 11-6)

■ Which of the following series are **<u>convergent</u>**:

(A) $\sum_{n=1}^{\infty}\frac{\sin 4n}{4^n}$. (B) $\sum_{n=1}^{\infty}\frac{1}{n+n\cos^2 n}$.

(C) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$. (D) $\sum_{n=1}^{\infty}\frac{1}{n^{\sin 1}}$.

Ans：AC

SOL：

(A). Let $a_n=\frac{\sin 4n}{4^n}$, $\lim_{n\to\infty}\left|\frac{\frac{\sin 4(n+1)}{4^{n+1}}}{\frac{\sin 4n}{4^n}}\right|=\lim_{n\to\infty}\left|\frac{\sin 4(n+1)}{4\sin 4n}\right|=\frac{1}{4}<1$.

By the ratio test, $\sum_{n=1}^{\infty}\frac{\sin 4n}{4^n}$ converges. <u>(11-6)</u>

(B). Let $a_n=\frac{1}{n+n\cos^2 n}$. $\frac{1}{n+n\cos^2 n}\geq\frac{1}{2n}$ and $\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges.

By the comparison test, $\sum_{n=1}^{\infty}\frac{1}{n+n\cos^2 n}$ diverges. <u>(11-4)</u>

(C). Let $a_n=\frac{\ln n}{n}$. $a_n>0$ satisfies (i) $a_{n+1}<a_n$ and (ii) $\lim_{n\to\infty}\frac{\ln n}{n}=0$.

<Pf.> (i). $\frac{d}{dx}\left(\frac{\ln x}{x}\right)=\frac{1-\ln x}{x^2}<0$ for x large.

By the Alternating series test, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converges. <u>(11-5)</u>

(D). Let $a_n=\frac{1}{n^{\sin 1}}$. By $p-$series, $\sum_{n=1}^{\infty}\frac{1}{n^{\sin 1}}$ diverges because $\sin 1\leq 1$. <u>(11-3)</u>

■ Which of the following statements are **TRUE**:

(A) If $\lim_{n\to\infty} a_n = 0$, where $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(B) If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then $\lim_{n\to\infty} |a_n| = 0$.

(C) If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then there exists a rearrangement $\{b_n\}_{n=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} b_n = \infty$.

(D) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Ans：BC

SOL：

(A). False

Given $a_n = \frac{1}{n}$, then $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} a_n$ diverges. (11-2)

(B). True

$\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent. Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} |a_n| = 0$. (11-6)

(C). True

Let $b_n^+ = \frac{a_n + |a_n|}{2}$ if $a_n > 0$ and $b_n^- = \frac{a_n - |a_n|}{2}$ if $a_n \le 0$. $b_n = b_n^+ + b_n^- = a_n$.

Since $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, $\sum_{n=1}^{\infty} |a_n|$ diverges.

Claim $\sum_{n=1}^{\infty} b_n^+$ diverges and $\sum_{n=1}^{\infty} b_n^-$ diverges.

(i) If $\sum_{n=1}^{\infty} b_n^+$ converges, then $\sum_{n=1}^{\infty} \frac{a_n + |a_n|}{2} - \sum_{n=1}^{\infty} \frac{a_n}{2} = \sum_{n=1}^{\infty} \frac{|a_n|}{2}$ converges. $(\rightarrow\leftarrow)$.

(ii) If $\sum_{n=1}^{\infty} b_n^-$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{2} - \sum_{n=1}^{\infty} \frac{a_n - |a_n|}{2} = \sum_{n=1}^{\infty} \frac{|a_n|}{2}$ converges. $(\rightarrow\leftarrow)$.

Hence, $\sum_{n=1}^{\infty} b_n$ diverges. (11-6)

(D).

By definition, if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent. So there is no $\sum_{n=1}^{\infty} a_n$ can be both absolutely convergent and conditionally convergent. (11-6)

■ Let a_n , b_n be sequences of real numbers. Which of the following statements must be true？

(A) If $\sum_n |a_n|$ converges, then $\sum_n a_n$ converges.

(B) If $\sum_n a_n$ converges, then $\sum_n (-1)^n |a_n|$ converges.

(C) If $\sum_n a_n$ and $\sum_n b_n$ are convergent, then $\sum_n (a_n + b_n)$ is convergent.

(D) If $\sum_n a_n$ and $\sum_n b_n$ are convergent, then $\sum_n (a_n b_n)$ is convergent.

Ans：AC