Ch11-8

單選題

■ Find the interval of convergence of $\sum_{n=0}^{\infty}\frac{(-3x)^n}{3n+1}$.

(a) $\left(\frac{-1}{3},\frac{1}{3}\right)$ (b) $\left(\frac{-1}{3},\frac{1}{3}\right]$ (c) $(-3,3)$ (d) $(-3,3]$.

Ans：b

Sol：

$$a_n=\frac{(-3x)^n}{3n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-3x)^{n+1}}{3n+4}\cdot\frac{3n+1}{(-3x)^n}\right|=|-3x|\lim_{n\to\infty}\left|\frac{3n+1}{3n+4}\right|=|-3x|$$

By the Ratio Test,the series $\sum_{n=0}^{\infty}\frac{(-3x)^n}{3n+1}$ converges when $|-3x|<1\Rightarrow -\frac{1}{3}<x<\frac{1}{3}$

When $x=\frac{1}{3}\Rightarrow\sum_{n=0}^{\infty}\frac{\left(-3\cdot\frac{1}{3}\right)^n}{3n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}$ converge

by Alternating series Test $\left(\because\lim_{n\to\infty}\frac{1}{3n+1}=0\right)$

When $x=-\frac{1}{3}\Rightarrow\sum_{n=0}^{\infty}\frac{\left(-3\cdot-\frac{1}{3}\right)^n}{3n+1}=\sum_{n=0}^{\infty}\frac{1}{3n+1}$ diverge

by Integral Test $\left(\because\int_0^{\infty}\frac{1}{3x+1}dx\quad diverge\right)$

the interval of convergence is $\left(-\frac{1}{3},\frac{1}{3}\right]$

■ Find the radius of convergence for the series $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

(a) ∞, (b) 1, (c) 2, (d) 0.

Ans：a

Sol：

$$a_n=\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{2^{2n+2}(n+1!)^2}\bullet\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|=\frac{x^2}{4}\lim_{n\to\infty}\left|\frac{1}{(n+1)^2}\right|=0$$

Thus,by the Ratio Test the series converge for all real x and we have $R=\infty$

■ Find the values of x for which the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}3^n}(x-1)^n$ converges.

A) $-3<x\le 3$ B) $-3\le x\le 3$ C) $-2<x\le 4$ D) $-2\le x\le 4$

Ans：C

Sol：

$$a_n=\frac{(-1)^n(x-1)^n}{\sqrt{n}\times 3^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{\sqrt{n+1}\times 3^{n+1}}\bullet\frac{\sqrt{n}\times 3^n}{(-1)^n(x-1)^n}\right|=\frac{|x-1|}{3}\lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{n+1}}\right|=\frac{|x-1|}{3}$$

By the Ratio Test,the series $\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{\sqrt{n}\times 3^n}$ converges when $\frac{|x-1|}{3}<1\Rightarrow -2<x<4$

When $x=4\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{\sqrt{n}\times 3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converge

by Alternating series Test $\left(\because\lim_{n\to\infty}\frac{1}{\sqrt{\text{n}}}=0\right)$

When $x=-2\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n(-3)^n}{\sqrt{n}\times 3^n}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverge

by Integral Test $\left(\because\int_1^{\infty}\frac{1}{\sqrt{x}}dx\quad diverge\right)$

the interval of convergence is $(-2,4]$

■ Find a value of b that will make the radius of convergence of power series

$$\sum_{n=2}^{\infty} \frac{b^n x^n}{\ln n}$$ equal to 5.

(a) 1. (b) $\frac{1}{5}$. (c) 5. (d) 10.

Ans：b

Sol：

$$a_n = \frac{b^n x^n}{\ln n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{b^{n+1}x^{n+1}}{\ln n+1} \bullet \frac{\ln n}{b^n x^n}\right| = |bx|\lim_{n\to\infty}\left|\frac{\ln n}{\ln n+1}\right| = |bx|$$

By the Ratio Test,Let $|bx| < 1 \Rightarrow -1 < bx < 1 \Rightarrow \frac{-1}{b} < x < \frac{1}{b} \Rightarrow \because r = 5 \therefore b = \frac{1}{5}$

■ Let $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. Let I, J, and K be the intervals of convergence for f, f', and f'', respectively. Which one contains only one end point?

A) I; B) J; C) K; D) none.

Ans：B

Sol：

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2},\ a_n=\frac{x^n}{n^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\bullet\frac{n^2}{x^n}\right|=\lim_{n\to\infty}\left|x\frac{n^2}{(n+1)^2}\right|=|x|$$

When $x=1\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}$ converge by Integral Test $\left(\because\int_1^{\infty}\frac{1}{x^2}dx=1\right)$

When $x=-1\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converge by Alternating series Test $\left(\because\lim_{n\to\infty}\frac{1}{n^2}=0\right)$

the interval of convergence is $[-1,1]$

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n},\ a_n=\frac{x^{n-1}}{n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^n}{n+1}\bullet\frac{n}{x^{n-1}}\right|=|x|$$

When $x=1\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n}$ diverge by Integral Test $\left(\because\int_1^{\infty}\frac{1}{x}dx=\infty\right)$

When $x=-1\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converge by Alternating series Test $\left(\because\lim_{n\to\infty}\frac{1}{n}=0\right)$

the interval of convergence is $[-1,1)$

$$f''(x)=\sum_{n=2}^{\infty}\frac{(n-1)x^{n-2}}{n},\ a_n=\frac{(n-1)x^{n-2}}{n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{nx^{n-1}}{n+1}\bullet\frac{n}{(n-1)x^{n-2}}\right|=|x|$$

When $x=1\Rightarrow\sum_{n=1}^{\infty}\frac{n-1}{n}$ diverge by Integral Test $\left(\because\int_1^{\infty}1-\frac{1}{x}dx=\infty\right)$

When $x=-1\Rightarrow\sum_{n=1}^{\infty}\frac{(n-1)(-1)^{n-2}}{n}$ diverge by Alternating series Test $\left(\because\lim_{n\to\infty}\frac{n-1}{n}=1\right)$

the interval of convergence is $(-1,1)$

Note: by test of diverge

■ For which α is the interval of convergence of

$$\sum_{n=1}^{\infty}\frac{n^{\alpha}}{n+1}x^{n}$$

equal to $[-1,1)$.

(A) 1; (B) 1/2; (C) -1/2; (D) -1.

Ans：B

Sol：

$$a_n=\frac{n^{\alpha}}{n+1}x^{n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^{\alpha}x^{n+1}}{n+2}\bullet\frac{n+1}{n^{\alpha}x^{n}}\right|=|x|$$

(1) When $x=1\Rightarrow\sum_{n=1}^{\infty}\frac{n^{\alpha}}{n+1}$ if it is diverge ,then $\alpha=1\text{or}\frac{1}{2}$ by Integral test

Note: by p-series and comparison

(2)When $x=-1\Rightarrow\sum_{n=1}^{\infty}\frac{n^{\alpha}}{n+1}(-1)^{n}$ if it is converge

,then $\lim_{n\to\infty}\frac{n^{\alpha}}{n+1}=0\Rightarrow\alpha=\frac{1}{2}or\frac{-1}{2}or-1$

Note: by Alternating series Test

By(1)&(2) ans(b)

■ The interval of convergence of $\sum_{n=1}^{\infty}\frac{(-1)^n(x+1)^n}{2^n n}$ is

(A) $\left(-3/2,-1/2\right]$, (B) $\left[-3/2,-1/2\right)$,

(C) $\left(-3,1\right]$, (D) $\left[-3,1\right)$.

Ans：C

Sol：

$$a_n=\frac{(-1)^n(x+1)^n}{n\times 2^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+1)^{n+1}}{(n+1)\times 2^{n+1}}\bullet\frac{n\times 2^n}{(-1)^n(x+1)^n}\right|=\frac{|x+1|}{2}\lim_{n\to\infty}\left|\frac{n}{n+1}\right|=\frac{|x+1|}{2}$$

By the Ratio Test,the series $\sum_{n=1}^{\infty}\frac{(-1)^n(x+1)^n}{n\times 2^n}$ converges when $\frac{|x+1|}{2}<1\Rightarrow -3<x<1$

When $x=1\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{n\times 2^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converge

by Alternating series Test $\left(\because \lim_{n\to\infty}\frac{1}{n}=0\right)$

When $x=-3\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n(-2)^n}{n\times 2^n}=\sum_{n=1}^{\infty}\frac{1}{n}$ diverge

by Integral Test $\left(\because \int_1^{\infty}\frac{1}{x}dx \quad diverge\right)$

the interval of convergence is $(-3,1]$

填空題

■ Find the interval of convergence for the power series $\sum_{k=1}^{\infty}\frac{(-1)^k}{5^k\cdot k}\cdot(x-2)^k$. ____________.

Ans： $(-3,7]$

Sol：

Let $a_k=\frac{(-1)^k}{5^k\cdot k}\cdot(x-2)^k$. Then

$$\left|\frac{a_{k+1}}{a_k}\right|=\left|\frac{(-1)^{k+1}(x-2)^{k+1}}{5^{k+1}\cdot(k+1)}\cdot\frac{5^k\cdot k}{(-1)^k(x-2)^k}\right|=\left|\frac{(-1)(x-2)k}{5\cdot(k+1)}\right|\to\left|\frac{x-2}{5}\right| \text{ as } k\to\infty$$

By the Ratio Test, the given series converges if $\left|\frac{x-2}{5}\right|<1$.

Thus the series converges in $(-3,7)$, but we must test for the endpoints of this interval.

For $x=-3$

$\sum_{k=1}^{\infty}\frac{(-1)^k}{5^k\cdot k}\cdot(-5)^k=\sum_{k=1}^{\infty}\frac{1}{k}$ which diverges(It's a harmonic series).

For $x=7$

$\sum_{k=1}^{\infty}\frac{(-1)^k}{5^k\cdot k}\cdot 5^k=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ which converges by the Alternating Series Test.

Therefore the given power series converges in the interval $(-3,7]$.

■ The power series $\sum_{n=1}^{\infty}a_n(x-2)^n$ and $\sum_{n=1}^{\infty}b_n(x-3)^n$ both converge at $x=6$. Find the largest interval over which both series must converge.

Ans： $0<x\le 6$

Sol：

$\sum_{n=1}^{\infty}b_n(x-3)^n$ 的收斂半徑為 R_a， $\sum_{n=1}^{\infty}a_n(x-2)^n$ 的收斂半徑為 R_b.

所以 $\begin{cases}|x-2|<R_a\Rightarrow 2-R_a<x<2+R_a\\ |x-3|<R_b\Rightarrow 3-R_b<x<3+R_b\end{cases}$,

為了使區間端點為 $x=6$，取 $R_a=4, R_b=3$，交集為 $0<x<6$

又因為兩個級數皆在 $x=6$ 處收斂，所以為 $0<x\le 6$

■ Find the *interval of convergence* of the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n3^n}$. ____(2)____

Ans：[-5,1)

Sol：

Let $a_n = \frac{(x+2)^n}{n \cdot 3^n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+2)^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x+2)^n}\right| = \left|\frac{(x+2)n}{3\cdot(n+1)}\right| \to \left|\frac{x+2}{3}\right| \text{ as } n \to \infty$$

By the Ratio Test, the given series converges if $\left|\frac{x+2}{3}\right| < 1$.

Thus the series converges in $(-5,1)$, but we must test for the endpoints of this interval.

For $x = -5$

$\sum_{k=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test.

For $x = 1$

$\sum_{n=1}^{\infty} \frac{(3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges(It's a harmonic series).

Therefore the given power series converges in the interval $[-5,1)$.

■ The interval of convergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{(x+2)^n}{n2^n}$ is ___(2)___.

Ans： $(-4,0]$

Sol：

Let $a_n = \frac{(-1)^n}{n\cdot 2^n}\cdot(x+2)^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)\cdot 2^{n+1}}\cdot\frac{n\cdot 2^n}{(-1)^n(x+2)^n}\right| = \left|\frac{(-1)(x+2)n}{2\cdot(n+1)}\right| \to \left|\frac{x+2}{2}\right| \text{ as } n\to\infty$$

By the Ratio Test, the given series converges if $\left|\frac{x+2}{2}\right|<1$.

Thus the series converges in $(-4,0)$, but we must test for the endpoints of this interval.

For $x=-4$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^n}\cdot(-2)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ which diverges(It's a harmonic series).

For $x=0$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 2^n}\cdot 2^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges by the Alternating Series Test.

Therefore the given power series converges in the interval $(-4,0]$.