Ch11-9

單選題

■ Let $f(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}x^n$. Find the interval of convergence for $f'(x)$.

A) $\boxed{(-1,1)}$ B) $\boxed{[-1,1)}$ C) $\boxed{(-1,1]}$ D) $\boxed{[-1,1]}$

Ans：B

Sol：

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n},\ a_n=\frac{x^{n-1}}{n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^n}{n+1}\bullet\frac{n}{x^{n-1}}\right|=|x|$$

When $x=1\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n}$ diverge by Integral Test $\left(\because\int_1^{\infty}\frac{1}{x}dx=\infty\right)$

When $x=-1\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converge by Alternating series Test $\left(\because\lim_{n\to\infty}\frac{1}{n}=0\right)$

the interval of convergence is $[-1,1)$

■ Suppose that the function $f(x)$ has the following power series representation , $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$, which of the following is NOT TRUE ?

(a) The radius of convergence is ∞.

(b) $f'(0)=0$.

(c) $f''(0)=-\frac{1}{4}$.

(d) f satisfies $x^2 f''+xf'+x^2 f=0$.

Ans：c

Sol：

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n},\ a_n=\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{-1}{(n+1)^2}\bullet\frac{x^2}{2^2}\right|=0<1 \qquad \forall x\in\Re$$

(a) is true

$$f'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n\bullet 2n\bullet x^{2n-1}}{(n!)^2\bullet 2^{2n}},$$

$$\because\ f'(x)=\frac{-2}{4}x+\frac{4}{64}x^3+\cdots\Rightarrow x=0,\ f'(0)=0$$

(b) is true

$$f''(x)=\frac{-2}{4}+\sum_{n=2}^{\infty}\frac{(-1)^n\bullet 2n\bullet(2n-1)x^{2n-2}}{(n!)^2\bullet 2^{2n}},$$

$$\because\ f''(x)=\frac{-2}{4}+\frac{12}{64}x^2+\cdots\Rightarrow x=0,\ f''(0)=\frac{-1}{2}$$

(c) is not true

$$x^2 f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+2}}{(n!)^2 2^{2n}}=x^2+\frac{-1}{4}x^4+\frac{-1}{64}x^6+\cdots$$

$$xf'(x)=\sum_{n=1}^{\infty}\frac{(-1)^n\bullet 2n\bullet x^{2n}}{(n!)^2\bullet 2^{2n}} = \frac{-2}{4}x^2+\frac{4}{64}x^4+\cdots$$

$$x^2f''(x)=\sum_{n=1}^{\infty}\frac{(-1)^n\bullet 2n\bullet (2n-1)x^{2n}}{(n!)^2\bullet 2^{2n}}=\frac{-2}{4}x^2+\frac{12}{64}x^4+\cdots$$

$$x^2f''+xf'+x^2f=0$$

(d) is true

Ans:(c)

填空題

■ Express $\ln 2$ as a convergent infinite series with all positive terms.___ (9)___.(11-9)

Ans： $\sum_{n=1}^{\infty}\frac{1}{n2^n}$

Sol：

$$-\ln(1-x)=\int\frac{1}{1-x}dx=\int(1+x+x^2+x^3+\cdots)dx=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+C$$

If we put $x=0$, $-\ln(1)=0=C$.

Then we put $x=\frac{1}{2}$, $-\ln(\frac{1}{2})=\ln 2=\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$

■ If $f(x)=\int_0^x \tan^{-1} t\,dt=\sum_{n=0}^{\infty} a_n x^n$ then $a_8=$ ______(1)______.

Ans： $-\dfrac{1}{56}$

Sol：

$$f'(x)=\frac{d}{dx}\int_0^x \tan^{-1} t\,dt=\tan^{-1} x\ \text{(微積分基本定理)}$$

$$\tan^{-1} x=\int \frac{1}{1+x^2}\,dx=\int\left(1-x^2+x^4-\cdots\right)dx=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+C$$

If we put $x=0$, $\tan^{-1} 0=0=C$.

$$f(x)=\int \tan^{-1} x=\int\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)=\frac{x^2}{2}-\frac{x^4}{4\cdot 3}+\frac{x^6}{6\cdot 5}-\cdots=\sum_{n=0}^{\infty} a_n x^n$$

Then $a_8=-\dfrac{1}{8\cdot 7}=-\dfrac{1}{56}$ (a_8 is the coefficient of x^8)