Ch12-5

單選題

Find the sine of the acute angle between two planes $x+y+z=0$ and

$x+2y+3z=1$.

A) $\boxed{\frac{\sqrt{3}}{\sqrt{21}}}$ B) $\boxed{\frac{36}{\sqrt{42}}}$ C) $\boxed{\frac{-\sqrt{3}}{\sqrt{21}}}$ D) $\boxed{\frac{-36}{\sqrt{42}}}$

Ans：A

SOL：

$n_1 = <1,1,1>, n_2 = <1,2,3>$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{6}{\sqrt{42}},$$

$$\Rightarrow \sin\theta = \sqrt{1-\left(\frac{6}{\sqrt{42}}\right)^2} = \sqrt{\frac{6}{42}} = \frac{\sqrt{3}}{\sqrt{21}}$$

(因為題目要求銳角，所以$\sin\theta$取正的。)

- Which of the following equation represents the plane that is tangent to both of the parametric

  curves $r_1(t) = \langle 3+t^2, 1-t, t\rangle$ and $r_2(\theta) = \langle \theta^2,\ \theta-2,\ 3-\theta\rangle$ at their

  intersection point?

  A) $y+z-1=0$ B) $x+z-1=0$ C) $x-3y-3z-1=0$

  D) $x+3y+3z-1=0$.

Ans：A

SOL：

先解出r_1, r_2交點

$$\begin{cases} 3+t^2=\theta^2 \\ 1-t=\theta-2 \\ t=3-\theta \end{cases} \Rightarrow \begin{cases} \theta=2 \\ t=1 \end{cases} \Rightarrow (x,y,z)=(4,0,1)$$

所以r_1, r_2在$(4,0,1)$的方向向量各別是

$\vec{v}_1 = <t,-1,1> = <1,-1,1>, v_2 = <\theta,1,-1> = <2,1,-1>$.

所以題目所求平面的法向量 $\vec{n} = v_1 \times v_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 0\hat{i} + 3\hat{j} + 3\hat{k} \Rightarrow \vec{n} =< 0,1,1 >$

所求平面 $L: y+z=c$，代入交點(4, 0, 1)，得 c=1.　$L: y+z-1=0$

■ Let L be the line passing through the point $(2,1,-1)$ and parallel to the line $\frac{1}{3}x+2=y-3=-\frac{1}{2}z$. Which of the following is NOT the equation of L?

A) $\begin{cases} x=3t+2 \\ y=t+1 \\ z=-2t-1 \end{cases}$

B) $\begin{cases} x-y+z=0 \\ 2y+z-1=0 \end{cases}$

C) $-2x+2=-6y+4=3z+1$

D) $x+y+2z=1$.

Ans：D

SOL：

$\frac{1}{3}x+2=y-3=-\frac{1}{2}z \Rightarrow \frac{x+6}{3}=\frac{y-3}{1}=\frac{z}{-2}$　$n =< 3,1,-2 >$

A) when t=0,(x,y,z)=(2,1,-1) and $n_1 =< 3,1,-2 > // n$

B) $n_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -3i - j + 2k, n_2 // n$, and (2,1,-1) pass the line.

C) $\frac{x-1}{-\frac{1}{2}} = \frac{y-\frac{2}{3}}{-\frac{1}{6}} = \frac{z+\frac{1}{3}}{\frac{1}{3}} \Rightarrow \frac{x-1}{-3} = \frac{y-\frac{2}{3}}{-1} = \frac{z+\frac{1}{3}}{2}$, so $n_3 =< -3,-1,2 > // n.$

And (2,1,-1) pass the line.

D) $x+y+2z=1$ 的法向量 $v =< 1,1,2 >$,

$\because v \cdot n = 3+1-4 = 0 \neq 1$　$\therefore$ 兩條線不平行.

Note: D is a plane ,not a line.

■ If θ is the angle between the planes $x-z=1$ and $2x+y-2z=3$, then $\cos\theta$ equals

A) $\frac{1}{\sqrt{2}}$ B) $\frac{4}{3\sqrt{2}}$ C) $\frac{2}{3\sqrt{2}}$ D) $\frac{1}{3\sqrt{2}}$.

Ans：B

SOL：

$v_1 = <1,0,-1>, v_2 = <2,1,-2>$. $\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{4}{3\sqrt{2}}$

■ **填空題**

■ Find the distance between two lines with parametric equations

$x=1+t,\quad y=-2+3t,\quad z=4-t$

and

$x=2s,\quad y=3+s,\quad z=-3+4s$

■ Ans：$\frac{8}{\sqrt{230}}$

SOL：

$v_1 = <1,3,-1>, v_2 = <2,1,4>$.

$\therefore$ a normal vector $n = v_1 \times v_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\hat{i} - 6\hat{j} - 5\hat{k}$.

Put s=0 ,we get the point (0,3,-3) on $L_2: x=2s,\quad y=3+s,\quad z=-3+4s$

And $L: 13(x-0)-6(y-3)-5(z+3)=0 \Rightarrow 13x-6y-5z+3=0$

Then let t=0, we get the point (1,-2,4) on $L_1: x=1+t,\quad y=-2+3t,\quad z=4-t$

Hence the distance between L_1 and L_2 is the same as the distance from (1,-2,4) to L. That is $D = \frac{|13\cdot 1 - 6(-2) - 5\cdot 4 + 3|}{\sqrt{13^2+(-6)^2+(-5)^2}} = \frac{8}{\sqrt{230}}$

■ Find the equation of the line of intersection of the two planes

$2x - y + z - 4 = 0$ and

$x + 3y - z - 2 = 0$. ____(1)____

Ans： $x = -2t, y = 3 + 3t, z = 7 + 7t, t \in \mathbf{R}$ **or** $\dfrac{x}{-2} = \dfrac{y-3}{3} = \dfrac{z-7}{7}$

SOL：

Let $v_1 = <2,-1,1>, v_2 = <1,3,-1>$. Since the vector v of the line L is orthogonal to the both v_1 and v_2. $v = v_1 \times v_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = -2\hat{i} + 3\hat{j} + 7\hat{k}$.

And let $x = 0$ in the equations of both planes. This gives the equations $2x - y + z - 4 = 0$ and $x + 3y - z - 2 = 0$,whose solution is $y = 3, z = 7$.

So the point (0,3,7) lies on L.

Hence $L: \dfrac{x}{-2} = \dfrac{y-3}{3} = \dfrac{z-7}{7}$