Ch13-3

單選題

■ Find the curvature of the helix $\overrightarrow{r(t)} = a\sin t\ \hat{i} + a\cos t\ \hat{j} + t\ \hat{k}$

(1) $\frac{1}{a}$ (2) $\frac{1}{a+1}$ (3) $\frac{a}{a^2+1}$ (4) $\frac{a}{a^2-1}$

Ans:3

Sol:

$r(t) = (a\sin t, a\cos t, \mathrm{t})$

$r'(t) = (a\cos t, -a\sin t, 1)$

$r''(t) = (\text{-}a\sin t, -a\cos t, 0)$

$$|r'(t)| = \sqrt{a^2\cos^2 t + a^2\sin^2 t + 1} = \sqrt{a^2+1}$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ a\cos t & -a\sin t & 1 \\ -a\sin t & -a\cos t & 0 \end{vmatrix} = (a\cos t, -a\sin t, -a^2\cos^2 t - a^2\sin^2 t) = (a\cos t, -a\sin t, -a^2)$$

$$|r'(t) \times r''(t)| = \sqrt{a^2\cos^2 t + a^2\sin^2 t + a^4} = a\sqrt{a^2+1}$$

$$\therefore k(t) = \frac{a\sqrt{a^2+1}}{(\sqrt{a^2+1})^3} = \frac{a}{a^2+1}$$

■ Find the length of the curve $r(t)=\left\langle \sin 2t, \cos 2t, 2t^{\frac{3}{2}} \right\rangle, 0 \le t \le 1$.

(a) $\frac{2}{27}\left(13\sqrt{13}-8\right)$ (b) $\frac{13}{9}$ (c) $\frac{13\sqrt{13}-6}{27}$ (d) $\frac{16}{9}$.

Ans：a

Sol:

$$r(t)=\left\langle \sin 2t, \cos 2t, 2t^{\frac{3}{2}} \right\rangle, 0 \le t \le 1 \Rightarrow r'(t)=\left\langle 2\cos 2t, -2\sin 2t, 3t^{\frac{1}{2}} \right\rangle$$

$$|r'(t)|=\sqrt{(2\cos 2t)^2+(-2\sin 2t)^2+\left(3t^{\frac{1}{2}}\right)^2}=\sqrt{4\cos^2 2t+4\sin^2 2t+9t}=\sqrt{4+9t}$$

$$L=\int_0^1 \sqrt{4+9t}\,dt=\frac{1}{9}\int_0^1 \sqrt{4+9t}\,d9t=\frac{1}{9}\cdot\frac{2}{3}(4+9t)^{\frac{3}{2}}\Big|_0^1=\frac{2}{27}\left(\sqrt{(13)^3}-8\right)$$

■ Let C be a curve described by $x=f(t),\ y=g(t),\ \alpha \le t \le \beta,$ where f' and g' are continuous on $[\alpha,\beta]$ and C is traversed exactly once as t runs from α to β. Which one of the following is always true?

(a) $\int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt \ge \beta-\alpha$;

(b) $\int_\alpha^\beta \sqrt{\left|\frac{dx}{dt}\right|+\left|\frac{dy}{dt}\right|}\,dt \ge \sqrt{\beta^2+\alpha^2}$;

(c) $\int_\alpha^\beta \sqrt{\left|\frac{dx}{dt}\right|+\left|\frac{dy}{dt}\right|}\,dt \ge \beta-\alpha$;

(d) $\int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt \ge \sqrt{(y(\beta)-y(\alpha))^2+(x(\beta)-x(\alpha))^2}$

Ans：d

Sol:

假設 x,y 皆為常數,則(a),(b),(c)左邊都為 0 右邊大於 0(不合)

而我們知道(d)的左邊代表弧長,右邊為直線距離,則當 C 為常數時兩者皆為 0

當 C 為直線兩者相等,C 為曲線弧長>直線距離 所以(d)正確

■ Let the distance traveled by a particle with position

$(x(t), y(t)) = (\sin^2 t, \cos^2 t)$ as t varies from $t = 0$ to $t = 3\pi$ be d.

Then $d =$

(a) $\sqrt{2}$. (b) 0. (c) $6\sqrt{2}$. (d) $4\sqrt{2}$.

Ans：c

Sol:

$$r(t) = (x(t), y(t)) = (\sin^2 t, \cos^2 t) \Rightarrow r'(t) = (2\sin t\cos t, -2\cos t\sin t)$$

$$|r'(t)| = \sqrt{(2\sin t\cos t)^2 + (-2\cos t\sin t)^2} = \sqrt{4\sin^2 t\cos^2 t + 4\cos^2 t\sin^2 t} = \sqrt{8}\sin t\cos t$$

$$L = 6\int_0^{\frac{\pi}{2}} \sqrt{8}\sin t\cos t dt = 6\sqrt{8}\cdot\frac{1}{2}(\sin t)^2\Big|_0^{\frac{\pi}{2}} = 6\sqrt{2}$$

■ The curvature of the cycloid.

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$$

at the point θ is.

(a) $\left|4a\cos\frac{1}{2}\theta\right|$. (b) $\frac{1}{|4a\cos\theta|}$.

(c) $\frac{1}{\left|4a\cot\frac{\theta}{2}\right|}$. (d) $\frac{1}{\left|4a\sin\frac{1}{2}\theta\right|}$.

Ans：d

Sol:

$$r(t) = (a(\theta - \sin\theta), a(1-\cos\theta), 0)$$
$$r'(t) = (a(1-\cos\theta), a\sin\theta, 0)$$
$$r''(t) = (a\sin\theta, a\cos\theta, 0)$$

$$|r'(t)| = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta} = \sqrt{2a^2(1-\cos\theta)}$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ a(1-\cos\theta) & a\sin\theta & 0 \\ a\sin\theta & a\cos\theta & 0 \end{vmatrix} = (0,0,a^2(\cos\theta - 1))$$

$$|r'(t) \times r''(t)| = \sqrt{a^4(\cos\theta - 1)^2} = a^2(1-\cos\theta)$$

$$\therefore k(t) = \frac{a^2(1-\cos\theta)}{(\sqrt{2a^2(1-\cos\theta)})^3} = \frac{1}{2\sqrt{2a^2(1-\cos\theta)}} = \frac{1}{2\sqrt{2a^2\left(1-(1-2\sin^2\frac{\theta}{2})\right)}} = \left|\frac{1}{2\cdot 2a\sin\frac{\theta}{2}}\right|$$

■ The principle normal vector $\vec{N}(t)$ for the curve $\vec{r}(t) = (t^3 - 3t)\vec{i} + 3t^2\vec{j}$ is.

(a) $\dfrac{t^2-1}{t^2+1}\vec{i} + \dfrac{2t}{t^2+1}\vec{j}$.

(b) $\dfrac{2t}{t^2+1}\vec{i} + \dfrac{1-t^2}{t^2+1}\vec{j}$.

(c) $\dfrac{t}{\sqrt{2t^2+2t+1}}\vec{i} + \dfrac{t+1}{\sqrt{2t^2+2t+1}}\vec{j}$.

(d) $\dfrac{t}{\sqrt{2t^2-2t+1}}\vec{i} + \dfrac{t-1}{\sqrt{2t^2-2t+1}}\vec{j}$.

Ans：b

Sol:

$$r'(t) = (3t^2-3)\vec{i} + 6t\vec{j}$$

$$\|r'(t)\| = \sqrt{(3t^2-3)^2 + (6t)^2} = 3(t^2+1)$$

$$T(t) = \frac{1}{3(t^2+1)}[(3t^2-3)\vec{i} + 6t\vec{j}] = \frac{1}{t^2+1}[(t^2-1)\vec{i} + 2t\vec{j}]$$

$$T'(t) = \left(\frac{t^2-1}{t^2+1}\right)'\vec{i} + \left(\frac{2t}{t^2+1}\right)'\vec{j} = \frac{4t}{(t^2+1)^2}\vec{i} + \frac{-2t^2+2}{(t^2+1)^2}\vec{j}$$

$$\|T'(t)\|\sqrt{\left(\frac{4t}{(t^2+1)^2}\right)^2 + \left(\frac{-2t^2+2}{(t^2+1)}\right)^2} = \frac{2}{t^2+1}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}\frac{\frac{4t}{(t^2+1)^2}}{\frac{2}{t^2+1}}\vec{i} + \frac{\frac{-2t^2+2}{(t^2+1)^2}}{\frac{2}{t^2+1}}\vec{j} = \frac{2t}{t^2+1}\vec{i} + \frac{-t^2+1}{t^2+1}\vec{j}$$

■ Find the unit tangent vector $\vec{T}(t)$ to the curve $\vec{r}(t) = (\sin t,\ 2t,\ t^2)$ when $t = 0$.

(1) $(\frac{-2}{\sqrt{5}},\ \frac{-1}{\sqrt{5}},\ 0)$ (2) $(\frac{2}{\sqrt{5}},\ \frac{-1}{\sqrt{5}},\ 0)$

(3) $(\frac{-2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0)$ (4) $(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0)$

(5) $(\frac{-1}{\sqrt{5}},\ \frac{-2}{\sqrt{5}},\ 0)$ (6) $(\frac{1}{\sqrt{5}},\ \frac{-2}{\sqrt{5}},\ 0)$

(7) $(\frac{-1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0)$ (8) $(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0)$

Ans：8

Sol:

$r'(t) = (\cos t, 2, 2t)$

$\|r'(t)\| = \sqrt{(\cos t)^2 + 4 + (2t)^2} = \sqrt{\cos^2 t + 4t^2 + 4}$

$T(t) = \frac{1}{\sqrt{\cos^2 t + 4t^2 + 4}}(\cos t, 2, 2t)$

$t = 0 \Rightarrow \cos t = 1 \therefore T(t) = \frac{1}{\sqrt{5}}(1,2,0)$

■ $\vec{r}(t) = (\cos t,\ \sin t,\ 5t)$, then the principal normal vector along $\vec{r}(t)$ at $t = 0$ is

(1) (0,1,0) (2) (-1,0,0) (3) (1,0,0) (4) (0,-1,0)

Ans：2

Sol:

$$r'(t) = (-\sin t, \cos t, 5)$$

$$\|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 25} = \sqrt{26}$$

$$T(t) = \frac{1}{\sqrt{26}}(-\sin t, \cos t, 5) \Rightarrow T'(t) = (\frac{-\cos t}{\sqrt{26}}, \frac{-\sin t}{\sqrt{26}}, 0)$$

$$\|T'(t)\|\sqrt{\left(\frac{-\cos t}{\sqrt{26}}\right)^2 + \left(\frac{-\sin t}{\sqrt{26}}\right)^2} = \frac{1}{\sqrt{26}}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = (\frac{\frac{-\cos t}{\sqrt{26}}}{\frac{1}{\sqrt{26}}}, \frac{\frac{-\sin t}{\sqrt{26}}}{\frac{1}{\sqrt{26}}}, 0) = (-\cos t, -\sin t, 0)$$

$$t = 0 \Rightarrow N(t) = (-1, 0, 0)$$

■ Let $r(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + e^t\mathbf{k}$. Find the unit tangent vector $T(t)$ at $t = 0$.

A) $\frac{1}{\sqrt{2}}(1,1,0)$ B) $\frac{1}{\sqrt{3}}(1,1,1)$ C) $\frac{1}{\sqrt{3}}(1,-1,1)$ D) $\frac{1}{\sqrt{2}}(1,-1,0)$

Ans：B

Sol:

$$r'(t) = (e^t \sin t + e^t \cos t)\vec{i} + (e^t \cos t - e^t \sin t)\vec{j} + e^t\vec{k}$$

$$\|r'(t)\| = \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 + e^{2t}} = \sqrt{3}e^t$$

$$T(t) = \frac{1}{\sqrt{3}e^t}[(e^t \sin t + e^t \cos t)\vec{i} + (e^t \cos t - e^t \sin t)\vec{j} + e^t\vec{k}] = \frac{1}{\sqrt{3}}[(\sin t + \cos t)\vec{i} + (\cos t - \sin t)\vec{j} + \vec{k}]$$

$$t = 0 \Rightarrow T(t) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

填空題

■ Find the arc length of the curve $x = 2t^2 - 5$, $y = 4t^2 + 11$, $0 \le t \le 4$.

Ans： $32\sqrt{5}$

Sol:

$$r(t) = (x(t), y(t)) = (2t^2 - 5, 4t^2 + 11) \Rightarrow r'(t) = (4t, 8t)$$

$$|r'(t)| = \sqrt{(4t)^2 + (8t)^2} = \sqrt{80}t$$

$$L = \int_0^4 \sqrt{80}t\,dt = \sqrt{80} \cdot \frac{1}{2}t^2 \Big|_0^4 = 8\sqrt{80} = 32\sqrt{5}$$

■ The displacement in meters of a particle moving in a straight line is given by $s(t) = t^2 + t$, where t is measured in seconds. Find the <u>average velocity</u> in meters per second over the time period [1, 4].

Ans：6

Sol:

$$\frac{s(4) - s(1)}{4 - 1} = \frac{20 - 2}{3} = 6$$

■ The arc length of the curve $(\cos t, \sin t, t)$ from $t = 0$ to $t = 4\pi$ is ___(2)___ .

Ans： $4\sqrt{2}\pi$

Sol:

$$r(t) = (x(t), y(t), z(t)) = (\cos t, \sin t, t) \Rightarrow r'(t) = (-\sin t, \cos t, 1)$$

$$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

$$L = \int_0^{4\pi} \sqrt{2}\,dt = \sqrt{2} \cdot t\Big|_0^{4\pi} = 4\sqrt{2}\pi$$

■ Find the curvature of the circle $x^2 + y^2 = 2003$ at $\left(\frac{\sqrt{2003}}{\sqrt{2}}, \frac{\sqrt{2003}}{2}\right)$.

Ans : $\frac{1}{\sqrt{2003}}$

Sol:

$$r(t) = (\sqrt{2003}\cos t, \sqrt{2003}\sin t, 0)$$

$$r'(t) = (-\sqrt{2003}\sin t, \sqrt{2003}\cos t, 0)$$

$$r''(t) = (-\sqrt{2003}\cos t, -\sqrt{2003}\sin t, 0)$$

$$|r'(t)| = \sqrt{2003}$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -\sqrt{2003}\sin t & \sqrt{2003}\cos t & 0 \\ -\sqrt{2003}\cos t & -\sqrt{2003}\sin t & 0 \end{vmatrix} = (0,0,2003) \quad |r'(t) \times r''(t)| = 2003$$

$$\therefore k(t) = \frac{2003}{(\sqrt{2003})^3} = \frac{1}{\sqrt{2003}}$$

■ Find the curvature of the curve $\mathbf{r}(t) = < t^2, t, t^3 >$ at $t = 1$. ___(2)___

Ans： $\frac{\sqrt{266}}{98}$

Sol:

$r(t) = (t^2, t, \mathrm{t}^3)$

$r'(t) = (2\mathrm{t}, 1, 3\mathrm{t}^2)$

$r''(t) = (2, 0, 6\mathrm{t})$

$$|r'(t)| = \sqrt{4t^2 + 1 + 9t^4}$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 2t & 1 & 3t^2 \\ 2 & 0 & 6t \end{vmatrix} = (6t, -6t^2, -2)$$

$$|r'(t) \times r''(t)| = \sqrt{(6t)^2 + (-6t^2)^2 + 4} = 2\sqrt{9t^2 + 9t^4 + 1}$$

$$\therefore k(t) = \frac{2\sqrt{9t^2 + 9t^4 + 1}}{(\sqrt{4t^2 + 1 + 9t^4})^3} \underset{t=1}{\Longrightarrow} \frac{2\sqrt{19}}{14\sqrt{14}} = \frac{\sqrt{266}}{98}$$