一百學年度第一學期微積分會考試題參考詳解

1.

(A)

$$\cos(4\pi x) + 0.5|_{x=0} = 1 + 0.5 = 1.5 \neq f(0) = 0.5.$$

(B) The function of the graph is an even function, but for the function $g(x) := \sin(4\pi x) + 0.5$,

$$g\left(\frac{1}{8}\right) = \sin\left(\frac{\pi}{2}\right) + 0.5 = 1.5 \neq g\left(-\frac{1}{8}\right) = \sin\left(-\frac{\pi}{2}\right) + 0.5 = -0.5.$$

Therefore, $\sin(4\pi x) + 0.5$ is NOT an even function.

(C) Let $f(x) = \sin(4\pi|x|) + 0.5$ for all $x \in \mathbb{R}$.

I. f is an even function;

II. $f(0) = 0.5$;

III. $f(x) = f(x + 0.5)$ for all $x \geq 0$;

IV.

$$\max_{0\leq x\leq 0.5} f(x) = f\left(\frac{1}{8}\right) = 1.5,$$

and

$$\min_{0\leq x\leq 0.5} f(x) = f\left(\frac{3}{8}\right) = -0.5.$$

All of the four properties are allowed for the function of the graph.

(D)

$$\cos(4\pi|x|) + 0.5|_{x=0} = 1 + 0.5 = 1.5 \neq f(0) = 0.5.$$

`Ans: C`

2.

$$\begin{aligned} & -0.6 < \sqrt{4x+5} - 3 < 0.6 \;\Rightarrow\; 2.4 < \sqrt{4x+5} < 3.6 \\ \Rightarrow\; & 5.76 < 4x + 5 < 12.96 \;\Rightarrow\; 0.76 < 4x < 7.96 \\ \Rightarrow\; & 0.19 < x < 1.99 \;\Rightarrow\; -0.81 < x - 1 < 0.99 \\ \Rightarrow\; & \delta \leq \min\{|-0.81|, |0.99|\} = 0.81. \end{aligned}$$

`Ans: C`

3.

$$-x^2 \le x^2 \cos\left(\frac{3}{x^4}\right) \le x^2,\ \lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0.$$

By Squeeze Theorem, $\lim_{x\to 0} x^2 \cos\left(\frac{3}{x^4}\right) = 0$.

`Ans: A`

4.

$$\begin{aligned} f^{-1}(x) &= \frac{1+x}{1-x}, \\ f(x) &= \frac{x-1}{x+1} = 1 - \frac{2}{x+1}. \\ f'(x) &= \frac{2}{(x+1)^2}. \\ f'(2) &= \frac{2}{(2+1)^2} = \frac{2}{9}. \end{aligned}$$

`Ans: D`

5. Way 1

$\pi \approx 3.14 > 2.71 \approx e$. (pick 1 such that $e - \pi + 1 > 0$.)

$$\begin{aligned} \lim_{x\to\infty} \left(\frac{\ln x}{x^e}\frac{x^\pi}{e^x}\right) &= \lim_{x\to\infty} \frac{\ln x}{x^{e-\pi+1}} \lim_{x\to\infty} \frac{x}{e^x} \left(\frac{\infty}{\infty} \ \& \ \frac{\infty}{\infty}\right) \\ &\overset{l'H}{=} \lim_{x\to\infty} \frac{1/x}{(e-\pi+1)x^{e-\pi}} \lim_{x\to\infty} \frac{1}{e^x} \\ &= \lim_{x\to\infty} \frac{1}{(e-\pi+1)x^{e-\pi+1}} \lim_{x\to\infty} \frac{1}{e^x} \\ &= 0 \cdot 0 = 0. \end{aligned}$$

Way 2

When $x \to \infty$,

$$1 < \ln x < x < x^2 < x^e < x^3 < x^\pi < x^4.$$

$$\frac{1}{e^x} = \frac{1}{x^3}\frac{x^3}{e^x} < \frac{\ln x}{x^e}\frac{x^\pi}{e^x} < \frac{x}{x^2}\frac{x^4}{e^x} = \frac{x^3}{e^x},$$

$$\lim_{x\to\infty} \frac{x^3}{e^x} \overset{l'H^3}{=} 0 = \lim_{x\to\infty} \frac{1}{e^x},$$

$$\lim_{x\to\infty} \left(\frac{\ln x}{x^e}\frac{x^\pi}{e^x}\right) = 0 \quad \text{(by squeeze theorem).}$$

`Ans: D`

6.

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx \overset{x=\sin\theta}{=} \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(1-\cos 2\theta)\,d\theta$$
$$= \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi/6} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.$$

`Ans: B`

7.

$$L = \int_1^4 \sqrt{1+(y')^2}\,dx = \int_1^4 \sqrt{1+(\sqrt{x^3-1})^2}\,dx$$
$$= \int_1^4 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big|_{x=1}^4 = \frac{64}{5} - \frac{2}{5} = \frac{62}{5}.$$

`Ans: A`

8. Notice that $0 < f(x) < 1$ for all $x \in (0,2)$. Hence $\int_1^2 f(x)\,dx > 0$,

$$\int_0^1 f(x)\,dx < \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx < \int_0^2 f(x)\,dx,$$

I < II. Now $0 < f^2(x) < f(x) < 1$ for all $x \in (0,1)$,

$$0 < \int_0^1 f^2(x)\,dx < \int_0^1 f(x)\,dx,$$

III < I. Finally, By fundamental theorem of calculus,

$$\int_0^2 f'(x)\,dx = f(2) - f(0) = 1 - 0 = 1 > \int_0^2 f(x)\,dx,$$

IV > II. In summary, III < I < II < IV.
`Ans: C`

9. Let $f(x) = \int_{1-x}^1 \dfrac{1}{t^3+t}\,dt$, then $\lim_{x\to 0^+} f(x) = 0$.
By the fundamental theorem of calculus,

$$f'(x) = \frac{1}{(1-x)^3 + (1-x)}.$$
$$I = \lim_{x\to 0^+} \frac{1}{x}\int_{1-x}^1 \frac{1}{t^3+t}\,dt$$
$$\overset{l'H}{=} \lim_{x\to 0^+} \frac{1}{(1-x)^3+(1-x)} = \frac{1}{2}.$$

Ans: D

10. Notice that $r^2 = 9\cos 5\theta = 0$ if and only if $\theta = \dfrac{(2k+1)\pi}{10}$, $k \in \mathbb{Z}$. Now, the area of one of the loops enclosed by the polar curve A is

$$\int_{-\pi/10}^{\pi/10} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/10}^{\pi/10} \frac{9}{2} \cos 5\theta \, d\theta = \frac{9}{10} \sin 5\theta \Big|_{-\pi/10}^{\pi/10} = \frac{9}{5}.$$

Ans: D

11.

(A)

$$(fg)'' = (f'g + fg')' = f''g + 2f'g' + fg''.$$

(B)

$$(fg' - f'g)' = f'g' + fg'' - f''g - f'g' = fg'' - f''g.$$

(C)

$$(\, f(g(x))\,)'' = (\, f'(g(x))g'(x)\,)' = f''(g(x))[g'(x)]^2 + f'(g(x))g''(x).$$

(D)

$$[f(x + g(\cos x))]' = f'(x + g(\cos x))(1 - g'(\cos x)\sin x).$$

Ans: BC

12.

(A) f is continuous everywhere since it is a rational function that the denominator is positive.

(B) $f(x) = x - \dfrac{\pi x}{x^2 + \pi}$, and $\lim_{x\to\pm\infty} \dfrac{-\pi x}{x^2+\pi} = 0$. Thus f has a slant asymptote $y = x$.

(C)

$$f''(x) = \frac{2\pi x(3\pi - x^2)}{(x^2+\pi)^3},$$

f has three inflection points $(0,0)$, $\pm\left(\sqrt{3\pi}, \dfrac{3\sqrt{3\pi}}{4}\right)$.

(D) f is differentiable on the whole real line.

$$f'(x) = 1 + \frac{\pi(x^2 - \pi)}{(x^2+\pi)^2} = 0$$

if and only if $x = 0$. But $f(-t) < f(0) = 0 < f(t)$ for any $t > 0$, $f(0)$ is neither maximum nor minimum. So f has neither maximum nor minimum.

Ans: AD

13.

(A)

$$\begin{aligned} i = 0,\ \frac{1}{2n} \cdot 4 = \frac{2}{n}; \\ i = n,\ \frac{1}{2n}\left(4 - \frac{3}{2}\right) = \frac{5}{4n}. \end{aligned}$$

(Observer the first term and the final term.)

$$\Delta x = \frac{1-0}{n} = \frac{1}{n},$$

$$\Delta x f(0) = \frac{1}{n} \cdot 2 = \frac{2}{n},\ \Delta x f(1) = \frac{1}{n} \cdot \frac{5}{4} = \frac{5}{4n}.$$

(B)

$$\begin{aligned} i = 0,\ \frac{1}{2n} \cdot 4 = \frac{2}{n}; \\ i = n,\ \frac{1}{2n}\left(4 - \frac{3}{2}\right) = \frac{5}{4n}. \end{aligned}$$

(Observer the first term and the final term.)

$$\Delta x = \frac{\frac{1}{2}-0}{n} = \frac{1}{2n},$$

$$\Delta x f(0) = \frac{1}{2n} \cdot 4 = \frac{2}{n},\ \Delta x f\left(12\right) = \frac{1}{2n} \cdot \frac{5}{2} = \frac{5}{4n}.$$

(C)

$$\begin{aligned} i = 0,\ \frac{1}{2n} \cdot 4 = \frac{2}{n}; \\ i = n,\ \frac{1}{2n}\left(4 - \frac{3}{2}\right) = \frac{5}{4n}. \end{aligned}$$

(Observer the first term and the final term.)

$$\Delta x = \frac{1-0}{n} = \frac{1}{n},$$

$$\Delta x f(0) = \frac{1}{n} \cdot 4 = \frac{4}{n},\ \Delta x f(1) = \frac{1}{n}(-2) = \frac{-2}{n}.$$

(D)

$$\begin{aligned} i = 0,\ \frac{1}{2n} \cdot 4 = \frac{2}{n}; \\ i = n,\ \frac{1}{2n}\left(4 - \frac{3}{2}\right) = \frac{5}{4n}. \end{aligned}$$

(Observer the first term and the final term.)

$$\Delta x = \frac{\frac{1}{2} - 0}{n} = \frac{1}{2n},$$

$$\Delta x f(0) = \frac{1}{2n} \cdot 2 = \frac{1}{n},\ \Delta x f\left(\tfrac{1}{2}\right) = \frac{1}{2n} \cdot \frac{29}{16} = \frac{5}{8n}.$$

`Ans: AB`

14. Notice that these curves intersect at $(-2, 2)$ and $(3, 3)$.

Way I: integrate with respect to variable x.

First, for $x \geq -6$,

$$\begin{aligned} y \geq 0 \ \text{ and } \ y^2 = x + 6 \Leftrightarrow y = \sqrt{x+6}; \\ y \leq 0 \ \text{ and } \ y^2 = x + 6 \Leftrightarrow y = -\sqrt{x+6}. \end{aligned}$$

Moreover, $\sqrt{x+6} \geq x$ while $-2 \leq x \leq 3$. Thus the **area** is

$$\int_{-6}^{-2} \left[\sqrt{x+6} - (-\sqrt{x+6})\right] \ dx + \int_{-2}^{3} \left(\sqrt{x+6} - x\right) \ dx$$

Way II: integrate with respect to variable y.

$y^2 - 6 \leq y$ while $-2 \leq y \leq 3$. Thus the **area** is

$$\int_{-2}^{3} \left[y - (y^2 - 6)\right] \ dy$$

`Ans: AD`

15.

(A)

$$\int_1^\infty \frac{1}{x} \ dx = \lim_{t \to \infty} \int_1^t \frac{1}{x} \ dx = \lim_{t \to \infty} \ln t = \infty$$

diverges.

(B)

$$\int_0^\infty \frac{1}{x^2+1} \ dx = \lim_{t \to \infty} \int_0^t \frac{1}{x^2+1} \ dx = \lim_{t \to \infty} \tan^{-1} t = \frac{\pi}{2}$$

converges.

(C)

$$\int_1^\infty e^{-x} \ dx = \lim_{t \to \infty} \int_1^t e^{-x} \ dx = \lim_{t \to \infty} \left(\frac{1}{e} - e^{-t}\right) = \frac{1}{e}.$$

converges.

(D) $0 < \dfrac{1}{x^4+x^2+1} < \dfrac{1}{x^4}$ for $x \geq 1$, $\int_1^\infty \dfrac{dx}{x^4} = \dfrac{1}{3}$ converges.

By Comparison Theorem, $\int_1^\infty \dfrac{1}{x^4+x^2+1}\, dx$ converges. Similarly, $\int_{-\infty}^{-1} \dfrac{1}{x^4+x^2+1}\, dx$ converges.

$\because \int_{-1}^{1} \dfrac{1}{x^4+x^2+1}\, dx$ is proper. $\int_{-\infty}^{\infty} \dfrac{1}{x^4+x^2+1}\, dx$ converges.

`Ans:` `BCD`

16. $\lim\limits_{x\to\frac{5}{2}^-} f'(x) = \lim\limits_{x\to\frac{5}{2}^-} (-x^2+4x-3)' = \lim\limits_{x\to\frac{5}{2}^-} (-2x+4) = -1.$

$\lim\limits_{x\to\frac{5}{2}^+} f'(x) = \lim\limits_{x\to\frac{5}{2}^-} (ax+b)' = \lim\limits_{x\to\frac{5}{2}^+} a = a. \implies a = -1.$

$\lim\limits_{x\to\frac{5}{2}^-} f(x) = \lim\limits_{x\to\frac{5}{2}^-} (-x^2+4x-3) = -\dfrac{25}{4} + 4\left(\dfrac{5}{2}\right) - 3 = \dfrac{3}{4}.$

$\lim\limits_{x\to\frac{5}{2}^+} f(x) = \lim\limits_{x\to\frac{5}{2}^+} (ax+b) = a\left(\dfrac{5}{2}\right) + b = b - \dfrac{5}{2}. \implies b = \dfrac{3}{4} + \dfrac{5}{2} = \dfrac{13}{4}.$

`Ans:` $\left(-1, \dfrac{13}{4}\right)$

17. $(\infty^0 \to \dfrac{\infty}{\infty})$

Let $y = (e^x + x)^{\frac{e}{x}}$, $\ln y = \dfrac{e}{x}\ln(e^x+x) = e\dfrac{\ln(e^x+x)}{x}$.

$$\begin{aligned}
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \left(\frac{e}{x}\ln(e^x+x)\right) \\
&= e\lim_{x\to\infty} \frac{\ln(e^x+x)}{x} \overset{l'H}{=} e\lim_{x\to\infty} \frac{(e^x+1)/(e^x+x)}{1} \\
&= e\lim_{x\to\infty} \frac{e^x+1}{e^x+x} \overset{l'H}{=} e\lim_{x\to\infty} \frac{e^x}{e^x+1} \overset{l'H}{=} e\lim_{x\to\infty} \frac{e^x}{e^x} = e.
\end{aligned}$$

$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^e.$

`Ans:` e^e

18. When $\theta = 0$, $x = 1$, $y = 1$, and the slope of the tangent line at $(1,1)$ is

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \left.\frac{\cos\theta - 2\sin 2\theta}{-\sin\theta + 2\cos 2\theta}\right|_{\theta=0} = \frac{1}{2}.$$

Thus the tangent line equation at $\theta = 0$ is

$$y = \frac{1}{2}(x-1) + 1 = \frac{1}{2}x + \frac{1}{2}.$$

`Ans:` $y = \dfrac{1}{2}(x-1) + 1 = \dfrac{1}{2}x + \dfrac{1}{2}.$

19.

$$S = \int 2\pi y \; ds$$

(separate into two parts, we can not do it from $0 \le \theta \le \pi$, $\because \theta = \dfrac{\pi}{2}$ is not smooth.)

$$= 2\int_0^{\pi/2} 2\pi \sin^3\theta \sqrt{(-3\cos^2\theta\sin\theta)^2 + (3\sin^2\theta\cos\theta)^2} \; d\theta$$

$$= 4\pi \int_0^{\pi/2} \sin^3\theta \cdot 3\sin\theta\cos\theta \; d\theta \overset{u=\sin\theta}{=} 12\pi \int_0^1 u^4 \; du = 12\pi \frac{u^5}{5}\Big|_0^1 = \frac{12}{5}\pi.$$

`Ans:` $\dfrac{12}{5}\pi$

20.

$$V = \int_{-1}^{0} \pi(x^3+1)^2 - \pi(x+1)^2 \; dx + \int_0^1 \pi(x+1)^2 - \pi(x^3+1)^2 \; dx$$

$$= \pi\left(\frac{1}{7}x^7 + \frac{1}{2}x^4 - \frac{1}{3}x^3 - x^2\Big|_{-1}^{0}\right) - \pi\left(\frac{1}{7}x^7 + \frac{1}{2}x^4 - \frac{1}{3}x^3 - x^2\Big|_0^1\right)$$

$$= \pi\left(\frac{1}{7} - \frac{1}{2} - \frac{1}{3} + 1\right) - \pi\left(\frac{1}{7} + \frac{1}{2} - \frac{1}{3} - 1\right) = \pi.$$

`Ans:` π