一百學年度第二學期微積分會考試題解答 (A 卷)

林峻緯

April 23, 2018

1. First, because $\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x}$, consider $x \in (-1, 1)$ then we have

$$\frac{-1}{1-x} = -\sum_{k=0}^{\infty} x^k.$$

Second, integrates two sides from last equation and let $k+1=n$, its implies

$$\ln(1-x) = -\sum_{k=0}^{\infty} \frac{x^{(k+1)}}{(k+1)} + C = -\sum_{n=1}^{\infty} \frac{x^n}{n} + C.$$

Finally, put $x = 0$ and we have $C = 0$, so

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}.$$

`Ans: (A).`

2. (A) Counterexample: $a_n = 0$, $b_n = -1$.
(B)(D) Counterexample: $a_n = 0$, $b_n = 1$.
(C) See the condition of the comparison test.
Note: Before use comparison test we need check two series which n-th terms have same sign ($a_n b_n \geq 0$) on every $n > N$.
`Ans: (C).`

3. (A) By p-series test, $p = 1.5 > 1$, it's convergent.
(B) Consider $n > 10$ (or large n such that $\sin(\pi/n) > 0$), by limit comparison test, set $b_n = \pi/n$ and $a_n = \sin(\pi/n)$ then we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} b_n$ is divergent, $\sum_{n=1}^{\infty} a_n$ is also divergent.
(C) By ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)/e^{n+1}}{n/e^n} \right| = \frac{1}{e} < 1,$$

so it's convergent.
(D) By root test,

$$\lim_{n\to\infty} \sqrt[n]{\left| \frac{n}{2n+1} \right|^n} = \frac{1}{2} < 1,$$

so it's convergent.
`Ans: (B).`

4. Consider $y = c \neq 0$ be fixed, then the intersection is a circle.
   (A) $z = x^2 - constant$ is a parabola.
   (B) $x^2 - z^2 = constant$ is a hyperbola.
   (C) $constant = x^2 + z^2$ is a circle.
   (D) $z^2 - x^2 = constant$ is a hyperbola.
   Note: In general, we consider
   (a) whether the graph contains origin. (b) the intersection of axes and the graph.
   (c) planes of intersection on its graph.
   `Ans: (C).`

5. $u(t)$ and $v(t)$ are perpendicular if and only if $|u(t)+v(t)|^2 = |u(t)-v(t)|^2$, so $t^2+3 = (t-1)^2$, hence $t = -1$
   `Ans: (B).`

6. (A) Consider two paths $y = 0$ and $x = 0$. For $y = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

   For $x = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1.$$

   it's not equal.
   (B) Consider two paths $y = 0$ and $x = 0$. For $y = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2-\sin^2 y}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

   For $x = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2-\sin^2 y}{x^2+y^2} = \lim_{y\to 0} \frac{-\sin^2 y}{y^2} = -1.$$

   it's not equal.
   (C) Consider two paths $y = 0$ and $x = 0$. For $y = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^3}{x^2} = 0.$$

   For $x = 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1.$$

   it's not equal.
   (D) Use triangle inequality, since

$$\left|\frac{x^3+y\sin^2 y}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y\sin^2 y}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2}\right| + \left|\frac{y\sin^2 y}{y^2}\right| = |x| + |y|\left|\frac{\sin^2 y}{y^2}\right|$$

   and $\lim_{(x,y)\to(0,0)} |x| + |y|\left|\frac{\sin^2 y}{y^2}\right| = 0 + 0\cdot 1 = 0$, by squeeze theorem, $\lim_{(x,y)\to(0,0)} \left|\frac{x^3+y\sin^2 y}{x^2+y^2}\right| = 0$ and it implies

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y\sin^2 y}{x^2+y^2} = 0$$

   `Ans: (D).`

7. Step1. (Find all critical points) Solve $\nabla f = 0$, we have $4x^3 - 16x = 0$ and $6y - 6 = 0$. That is, the critical points are $(x, y) = (0, 0), (2, 0), (-2, 0)$.
Step2. By second derivative test $\Delta(x, y) = f_{xx}f_{yy} - f_{yx}f_{xy} = (12x^2 - 16)(6) = 24(3x^2 - 4)$, we have $\Delta(0, 0) < 0$ and $\Delta(2, 0) = \Delta(-2, 0) > 0$. Step3. By step2 and Domain is $\mathbb{R}^2$, so (1) f has no absolute maximum. (2) f has two local minimum, one local maximum and no saddle point.
`Ans: (C).`

8.
$$\begin{aligned} y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} &= y\left(\frac{df(x^2-y^2)}{d(x^2-y^2)}\frac{\partial(x^2-y^2)}{\partial x}\right) + x\left(1 + \frac{df(x^2-y^2)}{d(x^2-y^2)}\frac{\partial(x^2-y^2)}{\partial y}\right) \\ &= y(f'(x^2-y^2)(2x)) + x(1 + f'(x^2-y^2)(-2y)) = x. \end{aligned}$$

`Ans: (B).`

9. Step1. the region of integration is a quarter circle in first quadrant.
Step2. Use polar coordinate: $x = r\cos\theta$ and $y = r\sin\theta$, then the question changes to be
$$\int_0^{\pi/2}\int_0^1 e^{-r^2} r\,dr\,d\theta = \frac{\pi}{2}\left[\frac{-e^{(-1)^2} + e^{0^2}}{2}\right] = \frac{\pi}{2}(1 - e^{-1}).$$
`Ans: (C).`

10. For $y^2 \le z \le 1$, $-1 \le y \le 1$, its region of x is $z \le x \le 1$. For another case $0 \le z \le y^2$, $1 \le y \le 1$, its region of x is $y^2 \le x \le 1$, so the solid can't present by 1 term. In fact,
$$\iiint_E f(x, y, z)dV = \int_{-1}^1\int_{y^2}^1\int_z^1 f(x, y, z)dxdzdy + \int_{-1}^1\int_0^{y^2}\int_{y^2}^1 f(x, y, z)dxdzdy.$$
`Ans: (D).`

11. (A) Use the fact $\lim_{n\to\infty}(1 + 1/n)^n = e$,
$$\lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n = \lim_{n\to\infty}\left(\left(1 + \frac{1}{n/2}\right)^{n/2}\right)^2 = \left(\lim_{n\to\infty}\left(1 + \frac{1}{n/2}\right)^{n/2}\right)^2 = e^2.$$
(B) Since $a_n = \ln\left(\frac{2n^2+1}{n^2+1}\right)$, $\lim_{n\to\infty} a_n = \ln\left(\lim_{n\to\infty}\frac{2n^2+1}{n^2+1}\right) = \ln 2$.
(C) It's loop for $4, 2, 1, \ldots$, so the sequence is divergent.
Note: $a_1 = 3$, $a_{n+1} = a_n/2$ if a_n is even, $a_{n+1} = 3a_n + 1$ if a_n is odd. It's known as $3n + 1$ *problem*.
(D) Use the fact $n! \gg 2^n$, where n is sufficient large, so the sequence is divergent.
`Ans: (A)(B).`

12. (A) Take $c_n = (-1/3)^n$, then the series converge at $x = 6$, but diverge at $x = 1$.
(B) By statement, it diverge at $x = 6$ implies the radius of convergence R satisfies $R \le 2$. Since $|1 - 4| = 3 > 2$, the series diverge at $x = 1$.
(C) The power series as same as original series has the same radius of convergence if we differentiate or integrate this power series.
(D) Choose $a_n = (1/3)^n$, $b_n = (1/2)^n$, then $A = 1/2$ and $B = 1$, but $\sum a_n b_n = \sum(1/6)^n = 1/5 \neq 1/2$
`Ans: (B)(C).`

13. (1) Consider $z = C > -1$ be fixed, then the trace is $(x+1)^2 + 2y^2 = constant > 0$ (ellipse).
(2) Consider $z = -1$, then the trace is $(x+1)^2 + 2y^2 = 0$ (point).
(3) Consider $z = C < -1$ be fixed, then the trace is $(x+1)^2 + 2y^2 < 0$ (No graph).
(4) Consider $y = C$ be fixed, then the trace is $z = (x+1)^2 + constant$ (parabola).
(5) Consider $x = C$ be fixed, then the trace is $z = 2y^2 + constant$ (parabola).
`Ans: (C)(D).`

14. (A) Consider two path: $x = y^3$ and $x = 0$, then for $x = y^3$,

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \frac{1}{2}.$$

For $x = 0$, $f(0,y) = 0$, so the limit is 0. Since $0 \neq \frac{1}{2}$, $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does not exist, f is discontinuous at (0,0).
(B)

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$$
$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{x} = \lim_{y\to 0} \frac{0}{y} = 0$$

(C) For $(a,b) \neq (0,0)$, we have

$$f_x(a,b) = \frac{b^3(a^2+b^6) - ab^3(2a)}{(a^2+b^6)^2} = \frac{b^3(b^6-a^2)}{(a^2+b^6)^2}.$$

(D) By (a) it's not continuous at $(0,0)$, so it's also not differentiable at $(0,0)$.
`Ans: (B)(C).`

15. (1) Let $x = au$, $y = bv$, $z = cw$, then the Jacobian is abc.
(2) By (1), we have

$$\iiint_E dV = \iiint_{u^2+v^2+w^2\leq 1} abc dV' = abc \iiint_{u^2+v^2+w^2\leq 1} dV' = abcS.$$

(3) By (2), S is a unit sphere, and we can use spherical coordinate: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$,

$$abcS = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^2\sin\phi d\rho d\theta d\phi = \int_0^{1}\int_0^{2\pi}\int_0^{\pi} \rho^2\sin\phi d\phi d\theta d\rho = \frac{4abc\pi}{3}$$

`Ans: (A)(C).`

1. For $x \in (-1,1)$,

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 + \frac{x}{2} + \frac{\frac{1}{2}\cdot\frac{-1}{2}}{2\cdot 1}x^2 + O(x^3),$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + O(x^3).$$

Hence the power series of $\frac{\sqrt{x+1}}{1-x}$ is

$$1 + \frac{3}{2}x + \frac{11}{8}x^2 + O(x^3).$$

`Ans:` $1 + \frac{3}{2}x + \frac{11}{8}x^2$.

2. (1) Compute the cross product:

$$\mathbf{u}(t) \times \mathbf{v}(t) = (3t - 2e^t, 2t - 3, e^t - t^2).$$

(2) compute the integral

$$\int_0^1 \mathbf{u}(t) \times \mathbf{v}(t) dt = \left(\int_0^1 (3t - 2e^t) dt, \int_0^1 (2t - 3) dt, \int_0^1 (e^t - t^2) dt \right)$$

.

$$= \left(\frac{3t^2}{2} - 2e^t, t^2 - 3t, e^t - \frac{t^3}{3} \right) \Big|_0^1 = \left(\frac{7}{2} - 2e, -2, e - \frac{4}{3} \right).$$

`Ans:` $(\frac{7}{2} - 2e, -2, e - \frac{4}{3})$.

3. (1) Let $F(x, y, z) = yz^3 + x^2z^2 - e^{xyz} = 0$, then use implicit differentiation, we have $F_z(x, y, z) = 3yz^2 + 2x^2z - e^{xyz}xy$ and $F_x(x, y, z) = 2xz^2 - e^{xyz}yz$.
(2)

$$\left. \frac{\partial z}{\partial x} \right|_{(x,y)=(0,1)} = -\frac{F_x(0, 1, 1)}{F_z(0, 1, 1)} = \frac{1}{3}.$$

`Ans:` $\frac{1}{3}$.

4. (method 1: Use Lagrange multiplier) Put $z = 1$ in two functions, and let $f_1(x, y) = f(x, y, 1) = x^2y+1$ and $g(x, y) = x^2 + y^2 - 9$, then let $\nabla f_1 = \lambda \nabla g$, we have

$$(2xy, x^2) = \lambda(2x, 2y).$$

Case1: If $x = 0$ and $\lambda \neq 0$, then $y = 0$, it contradicts the condition $g(0, 0) = -9 \neq 0$.
Case2: If $x = 0$ and $\lambda = 0$, we have $y = \pm 3$ and $f(0, \pm 3, 1) = 1$.
Case3: If $x \neq 0$, then we have $y = \lambda$ and $2\lambda^2 = x^2$, they imply $\lambda = \pm\sqrt{3}$. For $\lambda = \sqrt{3}$, we get absolute maximum $f(\pm\sqrt{6}, \sqrt{3}, 1) = 6\sqrt{3} + 1$. For $\lambda = -\sqrt{3}$, we have absolute minimum $f(\pm\sqrt{6}, -\sqrt{3}, 1) = -6\sqrt{3} + 1$.

(method 2: Use Arithmetic-Geometric Mean inequality.) We want to maximize $f_1(x, y) = f(x, y, 1) = x^2y + 1$ subjects to $x^2 + y^2 = 9$. First, it is obvious that f_1 has maximum on $y > 0$, so we can use this inequality:

$$3 = \frac{9}{3} = \frac{\frac{x^2}{2} + \frac{x^2}{2} + y^2}{3} \geq \sqrt[3]{\frac{x^2}{2} \cdot \frac{x^2}{2} \cdot y^2} = \sqrt[3]{\frac{x^4y^2}{4}}.$$

it implies $108 \geq x^4y^2$, that is, $6\sqrt{3} \geq x^2y$ and $6\sqrt{3} = x^2y$ iff $\frac{x^2}{2} = y^2 = 3$ ($y = \sqrt{3}$ and $x = \pm\sqrt{6}$).
`Ans:` $6\sqrt{3} + 1$.

5. $D = \{(r, \theta) \,|\, r \in [0, \sqrt{3}], \theta \in [0, \pi]\}$. Use the formula, we compute

$$m = \iint_D \rho(x, y)\, dA = \int_0^\pi \int_0^{\sqrt{3}} r^2 \sin\theta \, dr \, d\theta = 2\sqrt{3},$$

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x, y) dA = \frac{1}{m} \int_0^\pi \int_0^{\sqrt{3}} r^3 \sin\theta \cos\theta \, dr \, d\theta = \frac{1}{m} \int_0^\pi \sin\theta \cos\theta \, d\theta \int_0^{\sqrt{3}} r^3 \, dr = 0,$$

$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA = \frac{1}{m} \int_0^\pi \int_0^{\sqrt{3}} r^3 \sin^2\theta \, dr \, d\theta = \frac{9}{4 \cdot 2\sqrt{3}} \int_0^\pi \frac{1 - \cos 2\theta}{2} d\theta = \frac{9\pi}{16\sqrt{3}} = \frac{3\sqrt{3}\pi}{16}.$$

`Ans:` $(0, \frac{3\sqrt{3}\pi}{16})$.