一百零一學年度第一學期微積分會考試題參考詳解

1.

(A) Notice that $-|x|^3 \leq f(x) \leq |x|^3$.

As x approaches to 0, we have that $\lim_{x\to 0} f(x) = 0 = (0)$ by squeeze theorem.

(B) Notice that

$$0 \leq \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} \leq x^2.$$

As x tends to 0, we have that $f'(0) = \dfrac{f(x)-f(0)}{x-0} = 0.$

by squeeze theorem.

(C) Since both the rational numbers and irrational numbers are dense in the real line, for any partition, we always can pick the random point in every slice as rational numbers or irrational numbers.

Hence, the Riemann sum may be equal to 0 or approach

$$\int_0^1 x^3\,dx = \frac{1}{4}.$$

(Notice that the cubic function is integrable on [0,1].)

Thus f is NOT integrable on $[-1,1]$.

(D) Use the fact that π is an irrational number,

we can say this choice is correct.

`Ans:  C`

2.

$$f^{-1}(x) = \sin^{-1}(x - \frac{1}{2}),$$

$$(f^{-1})'(x) = \frac{1}{\sqrt{1-(x-1/2)^2}},$$

$$(f^{-1})'(0) = \frac{1}{\sqrt{1-(0-1/2)^2}} = \frac{2}{\sqrt{3}}.$$

Or

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{\cos(-\frac{\pi}{6})} = \frac{2}{\sqrt{3}}.$$

Ans: D

3.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{15}\frac{1}{n}=\int_0^1 x^{15}\,dx=\left.\frac{x^{16}}{16}\right|_0^1=\frac{1}{16}.$$

Ans: D

4. By fundamental theorem of calculus and l'Hopital's rule,

$$\lim_{x\to 0}\frac{\int_0^x \sin(t^2)\,dt}{x^2}\overset{l'H}{=}\lim_{x\to 0}\frac{\sin(x^2)}{2x}\overset{l'H}{=}\lim_{x\to 0}\frac{2x\cos(x^2)}{2}=0.$$

Ans: C

5.

(A) By deletion from (B), (C) and (D).

(B) For the parametric equation, $\max_{0\le t\le 1} y=\sin\left(2\pi\cdot\frac{1}{4}\right)=1$, but the maxima of y in graph is not equal to 1.

(C) For the parametric equation, $\max_{0\le t\le 1} y=\sin\left(2\pi\cdot\frac{1}{4}\right)=1$, but the maxima of y in graph is not equal to 1.

(D)

$$\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{-8t+4}{2\pi\cos(2\pi t)},$$

so $\left.\frac{dy}{dx}\right|_{t=0}=\frac{4}{2\pi}>0.$

Since $(x(0),y(0))=(0,0)$, the slope of tangent line of the curve at $(0,0)$ should be $\frac{4}{2\pi}$.

Since slope in the graph is 0, the graph does not match.

Ans: A

6.

(A)

$$\begin{aligned}\int_0^{\pi/2}\tan x\,dx&=\lim_{t\to\frac{\pi}{2}^-}\int_0^t\tan x\,dx\\&=\lim_{t\to\frac{\pi}{2}^-}\ln|\sec x|\big|_{x=0}^{t}=\lim_{t\to\frac{\pi}{2}^-}\ln|\sec t|=\infty.\end{aligned}$$

(B)

$$\begin{aligned}\int_0^{\infty} \tan^{-1} x \, dx &= \lim_{t\to\infty} \int_0^t \tan^{-1} x \, dx \\ &= \lim_{t\to\infty} \left[x\tan^{-1} x - \frac{1}{2}\ln(x^2+1) \right] \Bigg|_{x=0}^{t} \\ &= \lim_{t\to\infty} \left[t\tan^{-1} t - \frac{1}{2}\ln(t^2+1) \right] \\ &= \lim_{t\to\infty} \frac{1}{2} \ln\left(\frac{e^{t\tan^{-1} t}}{t^2+1} \right) = \frac{1}{2}\ln\left(\lim_{t\to\infty} \frac{e^{t\tan^{-1} t}}{t^2+1} \right) \overset{l'H}{=} \infty.\end{aligned}$$

(C)

$$\begin{aligned}\int_2^{\infty} \frac{1}{x\ln x} \, dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x\ln x} \, dx \\ &\overset{u=\ln x}{=} \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u} \, du = \lim_{t\to\infty} (\ln\ln t - \ln\ln 2) = \infty.\end{aligned}$$

(D)

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &\overset{x=\sin\theta}{=} \int_0^{\pi/2} \frac{d(\sin\theta)}{\sqrt{1-\sin^2\theta}} \\ &= \int_0^{\pi/2} d\theta = \frac{\pi}{2} - 0 = \frac{\pi}{2}.\end{aligned}$$

`Ans: D`

7. The **average** M is

$$\frac{1}{\frac{\pi}{2}-0} \int_0^{\pi/2} f(\theta) \, d\theta = \frac{4}{\pi} \tan\left(\frac{\theta}{2}\right) \Bigg|_{\theta=0}^{\pi/2} = \frac{4}{\pi}.$$

`Ans: B`

8. Notice that $r \geq 0$ for all $\theta \in \mathbb{R}$. The **area** A is

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} 2(1+\cos\theta)^2 \, d\theta = \int_0^{2\pi} 3 + 4\cos\theta + \cos 2\theta \, d\theta \\ &= 3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta \Bigg|_0^{2\pi} = 6\pi.\end{aligned}$$

`Ans: C`

9.

$$\begin{aligned}\int_0^{\pi/4} x\tan^2 x\ dx &= \int_0^{\pi/4} x(\sec^2 x - 1)\ dx\\ &= \int_0^{\pi/4} x\ d\tan x - \int_0^{\pi/4} x\ dx\\ &= x\tan x|_{x=0}^{\pi/4} - \int_0^{\pi/4} \tan x\ dx - \left.\frac{x^2}{2}\right|_{x=0}^{\pi/4}\\ &= \frac{\pi}{4} - \frac{\pi^2}{32} - \ln|\sec x||_{x=0}^{\pi/4} = \frac{\pi}{4} - \frac{\pi^2}{32} - \ln\sqrt{2}.\end{aligned}$$

`Ans: A`

10. Just find time $\tilde{t}$ such that the velocity $v(t)$ change sign at time $\tilde{t}$. $\tilde{t}$ should be equal to 2, 3, 5, 7 and 9, total five moments.
`Ans: B`

11.

$$\begin{aligned}&\lim_{x\to 0} y = \left(\lim_{x\to 0}\frac{1}{x-1}\right)\left(\lim_{x\to 0} x\sin\frac{1}{x}\right) = (-1)\cdot 0 = 0;\\ &\lim_{x\to 1^+} y = \infty, \quad \lim_{x\to 1^-} y = -\infty.\end{aligned}$$

So there is a vertical asymptote $x = 1$.

$$\lim_{x\to\pm\infty} y = \left(\lim_{x\to\pm\infty}\frac{x}{x-1}\right)\left(\lim_{x\to\pm\infty}\sin\frac{1}{x}\right) = 1\cdot 0 = 0.$$

So there is a horizontal asymptote $y = 0$.
`Ans: BC`

12.

$$F''(x) = x - \frac{4x}{(1+x^2)^2} = 0$$

when $x = 0, 1, -1$, and F is continuous and F'' changes sign at $0, 1, -1$.
`Ans: ABC`

13.

$$\begin{aligned}&-1 < \sqrt{x} - \sqrt{2} < 1,\\ \Leftrightarrow\ &\sqrt{2} - 1 < \sqrt{x} < \sqrt{2} + 1,\\ \Leftrightarrow\ &3 - 2\sqrt{2} < x < 3 + 2\sqrt{2},\\ \Leftrightarrow\ &1 - 2\sqrt{2} < x - 2 < 1 + 2\sqrt{2},\\ \Leftrightarrow\ &\delta = \min\{2\sqrt{2} - 1, 1 + 2\sqrt{2}\} = 2\sqrt{2} - 1 \approx 1.828.\end{aligned}$$

`Ans: BC`

14.

(A) Not the definition of improper integral.

(B) $\lim_{x\to\infty} \dfrac{1}{x} = 0$, and $\int_1^\infty \dfrac{1}{x}\,dx = \lim_{t\to\infty} \ln x|_{x=1}^{\infty} = \infty$.

(C)

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_0^\infty f(x)\,dx \\ &\text{(let } x=-t, dx=-dt) \\ &= \int_\infty^0 -f(-t)\,dt + \int_0^\infty f(x)\,dx \\ &= \int_0^\infty f(t)\,dt + \int_0^\infty f(x)\,dx = 2\int_0^\infty f(x)\,dx.\end{aligned}$$

(D) The comparison theorem for improper integral.
`Ans: CD`

15.

(A) $S = \int 2\pi y\ ds$, and the radius is $3 - f(x)$.

(B) $S = \int 2\pi x\ ds$, and the radius is $x - (-2) = 2 + x$.

(C)

$$P = \int_1^e \sqrt{1 + \left(\frac{2x}{x^2+1}\right)^2}\,dx,$$

$$Q = \int_1^e \sqrt{1 + \left(\frac{2}{x}\right)^2}\,dx.$$

Since

$$\frac{2x}{x^2+1} < \frac{2}{x} \quad \text{for } 1 \le x \le e,\ P \le Q.$$

(D) Let

$$f(x) = 2x,\ g(x) = 3, \quad 0 \le x \le 1.$$

Then $f(x) < g(x)$ on $[0,1]$, and

$$S = \int_0^1 \sqrt{1 + [f'(x)]^2}\,dx = \sqrt{5},$$

and

$$T = \int_0^1 \sqrt{1 + [g'(x)]^2}\,dx = 1.$$

`Ans: AB`

16. Let x be the distance between the car which travels west and the origin, y be the distance between the car which travels south and the origin, and z be the distance of two cars. Then $z^2 = x^2 + y^2$. Implicit differentiate it, $2zz' = 2xx' + 2yy'$, $x = x' \cdot t$, $y = y' \cdot t$, $z = z' \cdot t$.

$$t(z')^2 = t(x')^2 + t(y')^2,$$
$$z' = \sqrt{(x')^2 + (y')^2} = \sqrt{60^2 + 80^2} = 100.$$

`Ans:` 100

17. Let $u = x^2$. Then

$$g'(x) = \frac{d}{du} \int_1^u \sec(t-1)\, dt \frac{du}{dx} = \sec(u-1) \cdot \frac{du}{dx} = \sec(x^2 - 1) \cdot (2x).$$

So the linear approximation is $L(x) = g'(1)(x-1) + g(1) = \sec((1)^2 - 1) \cdot 2(1)(x-1) + 3 = 2x + 1$.

`Ans:` $L(x) = 2x + 1$

18.

$$\begin{aligned}
\int \frac{1}{(x^2+1)^2}\, dx &\overset{x=\tan\theta}{=} \int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta = \int \cos^2\theta\, d\theta \\
&= \int \frac{1}{2}(1 + \cos 2\theta)\, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C \\
&= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + C = \frac{\theta}{2} + \frac{1}{2}\frac{\tan\theta}{\sec\theta} + C \\
&= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{x^2+1} + C.
\end{aligned}$$

`Ans:` $\frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{x^2+1} + C$

19.

$$\begin{aligned}
L &= \int_0^\pi \sqrt{(-\sin t + \sin t + t\cos t)^2 + (\cos t - \cos t + t\sin t)^2}\; dt \\
&= \int_0^\pi t\; dt = \frac{t^2}{2}\Big|_0^\pi = \frac{\pi^2}{2}.
\end{aligned}$$

`Ans:` $\frac{\pi^2}{2}$

20.

$$\begin{aligned} &y = x^2 - 2x = 0 \;\Leftrightarrow\; x = 0, 2, \text{ the volume is} \\ &V = \int_0^2 2\pi x(0 - x^2 + 2x)\,dx = 2\pi \int_0^2 2x^2 - x^3\,dx \\ &\quad = 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4 \Big|_0^2 \right) = 2\pi \left(\frac{16}{3} - \frac{16}{4} \right) = \frac{8\pi}{3}. \end{aligned}$$

`Ans:` $\dfrac{8\pi}{3}$.