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1.

(A)

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+3}} = \frac{1}{27} \cdot \sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}.$$

Since it is a geometric sequence with ratio $\left|\frac{-2}{3}\right| < 1$, the series is convergent. And the series

$$\sum_{n=1}^{\infty} \left|\frac{(-2)^n}{3^{n+3}}\right| = \sum_{n=1}^{\infty} \frac{2^n}{3^{n+3}} = \frac{1}{27} \cdot \sum_{n=1}^{\infty} \frac{2^n}{3^n}$$

is a geometric sequence with ratio $\frac{2}{3}$. So the series is absolute convergent.

(B) Since $\sin\left(\frac{\pi}{n}\right)$ is decreasing to 0 as n goes to infinity, the alternative series is convergent by alternating series test. On the other hand, consider the series

$$\sum_{n=5}^{\infty} \left|(-1)^n \sin\left(\frac{\pi}{n}\right)\right| = \sum_{n=5}^{\infty} \sin\left(\frac{\pi}{n}\right).$$

Since

$$\lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 1,$$

the series $\sum_{n=5}^{\infty} \sin\left(\frac{\pi}{n}\right)$ converges if and only if the series $\sum_{n=5}^{\infty} \frac{\pi}{n}$ converges. And the series $\sum_{n=5}^{\infty} \frac{\pi}{n}$ is a p-series with $p = 1 \le 1$, so the series $\sum_{n=5}^{\infty} \frac{\pi}{n}$ diverges and the series $\sum_{n=5}^{\infty} \sin\left(\frac{\pi}{n}\right)$ diverges, too. Hence the series $\sum_{n=5}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)$ is conditionally convergent.

(C) Since

$$\lim_{\to\infty} e^{-\frac{1}{n+1}} = 1 \neq 0,$$

the series $\sum_{n=1}^{\infty}(-1)^n e^{-\frac{1}{n+1}}$ is diverges.

(D) Since $\sin(n\pi) = 0$, $\forall\, n \in \mathbb{N}$, the series

$$\sum_{n=4}^{\infty} \sin(n\pi) \sin\left(\frac{\pi}{n}\right) = 0.$$

Hence the series converges. On the other hand, the series

$$\sum_{n=4}^{\infty} \left|\sin(n\pi) \sin\left(\frac{\pi}{n}\right)\right| = 0.$$

is converges. Hence the series $\sum_{n=4}^{\infty} \sin(n\pi) \sin\left(\frac{\pi}{n}\right)$ is absolute convergent.

`Ans: B`

2.
Note that

$$(1+y)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} y^n, \quad |y|<1.$$

Set $\alpha = -\dfrac{1}{2}$ and $y=-x^2$, then we have

$$(1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$$

$$\Rightarrow \int_0^t \frac{1}{\sqrt{1-x^2}}dx = \int_0^t \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n dx = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^n}{2n+1} t^{2n+1}.$$

Next, set $t=\dfrac{1}{2}$, we have

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^n}{(2n+1)2^{2n+1}};$$

$$\text{left hand side} = \arcsin x \,\Big|_{x=0}^{\frac{1}{2}} = \frac{\pi}{6}.$$

$$\Rightarrow \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^n}{(2n+1)2^{2n+1}} = \frac{\pi}{6} \Rightarrow \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^{n+1}}{(2n+1)4^n} = (-2)\cdot\frac{\pi}{6} = -\frac{\pi}{3}.$$

`Ans: A`

3. From

$$(x(t),y(t),z(t)) = \left(e^t, te^t, te^{t^2}\right) = (1,0,0),$$

we have $t=0$.
Note that,

$$(x'(t),y'(t),z'(t)) = \left(e^t, e^t+te^t, e^{t^2}+2t^2e^{t^2}\right),$$

and

$$(x'(0),y'(0),z'(0)) = (1,1,1).$$

Hence the tangent line equation at the point $(1,0,0)$ is

$$x-1=y=z.$$

Let $x-1=y=z=t$, then $x=t+1$, $y=t$, and $z=t$. Substitute into this fraction $\dfrac{a+b-1}{c}$, where $a=x$, $b=y$, and $c=z$, then we have

$$\frac{t-1+t-1}{t} = 2.$$

`Ans: D`

4. The slope vector of L_1 is $(5,2,-1)$ and the slope vector of L_2 is $(0,-1,3)$. So L_1 and L_2 are not parallel and the same, either. Next, check whether these two lines have intersection points. Substitute $x=-1$ into L_1, we have $x=-1$, $y=2$, and

$z = 4$. And the point $(-1, 2, 4)$ lies on the line L_2, when $t = -1$. Hence L_1 and L_2 are intersecting.

`Ans: C`

5. Note that $f_x = 3x^2 + 2y$, $f_y = 2x - 2y$, $f_{xx} = 6x$, $f_{xy} = 2$, $f_{yx} = 2$, and $f_{yy} = -2$. From $\nabla f = (f_x, f_y) = (0, 0)$, we solve this equation

$$\begin{cases} 3x^2 + 2y & = 0, \\ 2x - 2y & = 0. \end{cases}$$

Then we get two critical points which are $(0, 0)$ and $\left(-\frac{2}{3}, -\frac{2}{3}\right)$. Since

$$f_{xx}(0,0) \cdot f_{yy}(0,0) - f_{xy}(0,0) \cdot f_{xy}(0,0) = -4 < 0,$$

the critical point $(0, 0)$ is a saddle point. On the other hand,

$$f_{xx}\left(-\frac{2}{3}, -\frac{2}{3}\right) \cdot f_{yy}\left(-\frac{2}{3}, -\frac{2}{3}\right) - f_{xy}\left(-\frac{2}{3}, -\frac{2}{3}\right) \cdot f_{xy}\left(-\frac{2}{3}, -\frac{2}{3}\right) = 4 > 0,$$

and

$$f_{xx}\left(-\frac{2}{3}, -\frac{2}{3}\right) = -4 < 0,$$

so the critical point $\left(-\frac{2}{3}, -\frac{2}{3}\right)$ is a local maximum.

`Ans: D`

6. (1) When $(x, y) = (t, t^2)$, then as $(x, y) \to (0, 0)$, $t \to 0$. Hence

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^4} = \lim_{t\to 0} \frac{t^4}{t^4 + t^8} = 1.$$

But, when $(x, y) = (0, t)$, then as $(x, y) \to (0, 0)$, $t \to 0$. Hence

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^4} = \lim_{t\to 0} \frac{0}{0 + t^4} = 0.$$

Since, with different direction, we have different limit value, this limit doesn't exist.
(2) Case 1. When $x = 0$, then the limit is 0, obviously.
Case 2. When $x \neq 0$, then

$$\left|\frac{x^2 y}{x^2 + y^4 + z^4}\right| \leq \left|\frac{x^2 y}{x^2}\right| = |y|,$$

and

$$\lim_{(x,y,z)\to(0,0,0)} y = 0.$$

By comparison theorem, the limit exists.

`Ans: C`

7. Let's take $f(x, y) = \begin{cases} x + y & , x = 0 \text{ or } y = 0 \\ 1 & , \text{else} \end{cases}$ as an example with $(a, b) = (0, 0)$.

Note that $f_x(0,0) = 1$ and $f_y(0,0) = 1$, obviously.
(A) When $(x, y) = (t, 0)$, then as $(x, y) \to (0, 0)$, $t \to 0$. Hence

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{t\to 0} t + 0 = 0.$$

But, when $(x, y) = (t, t)$, then as $(x, y) \to (0, 0)$, $t \to 0$. Hence

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{t\to 0} 1 = 1.$$

So $\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist.
(B) Since $\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist, f is not differentiable at $(0, 0)$.
(C) Choose $\vec{u} = (t, t)$, then

$$D_{\vec{u}} f(0, 0) = \lim_{t\to 0} \frac{f(t, t) - f(0, 0)}{t} = \lim_{t\to 0} \frac{1}{t}.$$

But $\lim_{t\to 0} \frac{1}{t}$ doesn't exist.
(D) f is not continuous at $(0, 0)$ by (A).
`Ans: D`

8. (A)

$$S = \{(x, y, z) \,|\, y = g(x, z)\} = \{(x, y, z) \,|\, 0 = g(x, z) - y\} = S_0$$

(B) The equation for T_0 is

$$(x - x_0, y - y_0, z - z_0) \cdot (G_x, G_y, G_z) = 0.$$

Note that

$$G_x(x_0, y_0, z_0) = g_x(x_0, z_0),$$
$$G_y(x_0, y_0, z_0) = -1,$$

and

$$G_z(x_0, y_0, z_0) = g_z(x_0, z_0).$$

So substitute into the equation, we have

$$g_x(x_0, z_0)(x - x_0) - (y - y_0) + G_z(x_0, y_0, z_0)(z - z_0) = 0.$$

(C) By (A), since $S = S_0$, $T = T_0$.
(D) By (B) and (C), the equation of T is

$$G_z(x_0, y_0, z_0)(x - x_0) - (y - y_0) + G_z(x_0, y_0, z_0)(z - z_0) = 0.$$

`Ans: B`

9. Let $f(x, y) = x^2$. Note that $f(x, y) = f(x, -y) = f(-x, y) = f(-x, -y)$. Hence

$$\iint_\Omega x^2 dA = 4 \cdot \int_0^1 \int_0^{1-x} x^2 \, dydx = 4 \cdot \int_0^1 (x^2 - x^3) \, dx = 4 \cdot \frac{1}{12} = \frac{1}{3}.$$

`Ans: A`

10. The volume of the region V is

$$V = \int_0^4 \int_0^{\frac{\sqrt{16-y^2}}{2}} \int_0^{4-y} dxdzdy$$

$$= \int_0^4 \int_0^{\frac{\sqrt{16-y^2}}{2}} (4-y)dzdy.$$

Let $y = 2r\cos\theta$, $z = r\sin\theta$, where $0 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$.

$$V = \int_0^{\frac{\pi}{2}} \int_0^2 (4 - 2r\cos\theta) \cdot 2r\, drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(16 - \frac{32\cos\theta}{3}\right) d\theta$$

$$= 8\pi - \frac{32}{3}.$$

`Ans: B`

11. Let write down some terms of this series.

$$a_0 = 2, a_1 = 3, a_2 = 4, a_3 = 4.$$

We find that the series converges to 4 by first three terms, and the series $\{2, 3, 4, 4, 4, \cdots\}$ is a increasing and bounded series, obviously.
`Ans: ABD`

12. (A) By ratio test, we want

$$\lim_{n\to\infty} \frac{\frac{|x-3|^{n+1}}{\sqrt{n+1}}}{\frac{|x-3|^n}{\sqrt{n}}} = \lim_{n\to\infty} |x-3| \frac{\sqrt{n}}{\sqrt{n+1}} = |x-3| < 1.$$

Then we have the domain of $g(x)$ includes $(2, 4)$.
When $x = 2$, we should check whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent. Since $\frac{1}{\sqrt{n}}$ is a decreasing series as n gets large, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent by alternative series test.
When $x = 4$, we we should check whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is convergent. By p-series test, we know the series is diverges. Hence the domain of $g(x)$ is $[2, 4)$.
(B) Since the domain of $g(x)$ is $[2, 4)$, $g(x)$ is differentiable on $(2, 4)$.
(C)

$$g'(x) = \sum_{n=1}^{\infty} \frac{n(x-3)^{n-1}}{\sqrt{n}}, \quad g'(3) = \frac{1}{\sqrt{1}} = 1.$$

(D)

$$g''(x) = \sum_{n=2}^{\infty} \frac{n(n-1)(x-3)^{n-2}}{\sqrt{n}}, \quad g''(3) = \frac{2\cdot(2-1)}{\sqrt{2}} = \sqrt{2}.$$

`Ans: BD`

13. (A) Since

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{1-1}{x} = 0,$$
$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{1-1}{y} = 0,$$

both $f_x(0,0)$ and $f_y(0,0)$ exist.
(B) When $(x,y)=(t,t)$, then as $(x,y)\to(0,0)$, $t\to 0$. Hence

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = \lim_{t\to 0} \frac{0}{2t^2} = 0.$$

But, when $(x,y)=(t,0)$, then as $(x,y)\to(0,0)$, $t\to 0$. Hence

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = \lim_{t\to 0} 1 = 1.$$

So f is not continuous at $(0,0)$.
(C) By (B), since f is not continuous at $(0,0)$, f is not differentiable at $(0,0)$.
(D) Since f is a rational function, and the denominator of f are not 0, when $(a,b)\neq(0,0)$, f is differentiable for $(a,b)\neq(0,0)$.
`Ans: AD`

14. Plot the region of each options.
`Ans: ABD`

15. Since the bounds of y and z in the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x}} f(x,y,z)\,dzdydx$ only depend on x, it doesn't need to change the bound while switching the arrangement of y and z. Hence (A) is wrong and (B) is correct.
From the three inequalities $0\le x\le 1$, $0\le y\le\sqrt{1-x^2}$, and $0\le z\le\sqrt{x}$, we have $0\le z\le 1$, $0\le y\le\sqrt{1-z^4}$, and $z^2\le x\le\sqrt{1-y^2}$. Hence (C) is correct, but (D)is wrong.
`Ans: BC`

16. Note that,

$$r'(t) = (-3\sin 3t, 3\cos 3t),$$

and

$$r(t) = (1,0) \Rightarrow t = 0.$$

Now, the arc length s measured from $(1,0)$ is

$$\int_0^t |r'(t)|\,dt = \int_0^t \sqrt{((-3\sin 3t)')^2 + ((-3\cos 3t)')^2}\,dt = \int_0^t 3\,dt = 3t.$$

Denote $3t$ with s, then we have

$$r(t(s)) = \cos si + \sin si.$$

Ans: $\cos s\mathtt{i} + \sin s\mathtt{j}$

17. Doing partial derivative on the equation $z^3x + (x^2+y)z = 0$, we have

$$3z^2 \cdot \frac{\partial z}{\partial x} \cdot x + z^3 + 2xz + (x^2+y)\frac{\partial z}{\partial x} = 0.$$

Then substitute $x=0$, $y=1$, and $z=0$ into the equation above. We have

$$0+0+0+\left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(0,1,0)} = 0$$

$$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(0,1,0)} = 0$$

Ans: 0

18. Since

$$b^{\ln(n^3+1)} = \left(n^3+1\right)^{\ln b},$$

$$\sum_{n=1}^{\infty} b^{\ln(n^3+1)} = \sum_{n=1}^{\infty} \left(n^3+1\right)^{\ln b}.$$

By limit comparison test, we have

$$\lim_{n\to\infty} \frac{(n^3+1)^{\ln b}}{(n^3)^{\ln b}} = 1,$$

so the series $\sum_{n=1}^{\infty}(n^3+1)^{\ln b}$ diverges if and only if $\sum_{n=1}^{\infty}(n^3)^{\ln b}$ diverges. Next, by p-series test, when $\ln b^3 > -1$, the series is divergent. Hence the minimum positive value of b is $e^{-\frac{1}{3}}$.

Ans: $e^{-\frac{1}{3}}$

19. By Fubini's theorem,

$$\iint_R \frac{\sin y}{y}\, dA = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dxdy = \int_0^{\pi} \sin y = 1-(-1) = 2.$$

Ans: 2

20. The distance from the origin to a point (x,y,z) is $\sqrt{x^2+y^2+z^2}$. To find the farthest point, it is equivalent to find the maximum of $x^2+y^2+z^2$. Let $f(x,y,z) = x^2+y^2+z^2$, $g(x,y,z) = x+y+2z-2$, and $h(x,y,z) = x^2+y^2-z$. Using Lagrange's multiplier, we should solve the equation for x, y, and z

$$\nabla f + \alpha\nabla g + \beta\nabla = (0,0,0)$$

$$\Rightarrow (2x,2y,2z) + (\alpha,\alpha,2\alpha) + (2\beta x, 2\beta y, -\beta) = (0,0,0).$$

Next, we solve the system of equations

$$\begin{cases} 2x+\alpha+2\beta x = 0 \\ 2y+\alpha+2\beta y = 0 \\ 2z+2\alpha-\beta = 0. \end{cases}$$

Then, we have two critical points which are $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(x, y, z) = (-1, -1, 2)$, and the distances are $\frac{3}{4}$ and 6, respectively.

`Ans: (-1,-1,2)`