一百零二學年度第一學期微積分會考試題參考詳解

1.

$$\begin{aligned}\left|e^{-x}-1\right|<\frac{1}{2} &\Rightarrow -\frac{1}{2}<e^{-x}-1<\frac{1}{2}\\ &\Rightarrow \frac{1}{2}<e^{-x}<\frac{3}{2}\\ &\Rightarrow -\ln 2=\ln\frac{1}{2}<-x<\ln\frac{3}{2}\\ &\Rightarrow -\ln\frac{3}{2}<x<\ln 2.\end{aligned}$$

Since

$$|\ln 2|>\left|\ln\frac{3}{2}\right|,$$

$$\delta=\min\left\{\ln 2,\,\ln\frac{3}{2}\right\}=\ln\frac{3}{2}.$$

`Ans: C`

2.

(A) $f'(x)=7-\sin x\geq 7-1>0$ for all $x\in\mathbb{R}$. Hence f is strictly increasing, f has at most one root. Since $f\left(-\frac{\pi}{2}\right)f(0)=\left(-\frac{7\pi}{2}\right)\cdot 1<0$, f has a root in $\left(-\frac{\pi}{2},0\right)$ by intermediate value theorem.

(B) $\lim_{x\to\infty}7x=\infty$ and $|\cos x|\leq 1$ for all $x\in\mathbb{R}$, so $\lim_{x\to\infty}f(x)=\infty$. Thus f is not periodic.

(C) $f'(x)=7-\sin x\geq 7-1>0$ for all $x\in\mathbb{R}$. Hence f is strictly increasing, f has at most one root. Since $f\left(-\frac{\pi}{2}\right)f(0)=\left(-\frac{7\pi}{2}\right)\cdot 1<0$, f has a root in $\left(-\frac{\pi}{2},0\right)$ by intermediate value theorem.

(D)

$$\lim_{x\to\pm\infty}|f(x)-(ax+b)|=\lim_{x\to\pm\infty}|(7-a)x+\cos x+b|\neq 0$$

for and $a,b\in\mathbb{R}$. Thus f has no any horizontal/ slant asymptote.

`Ans: C`

3. For $x \neq 0$,

$$\begin{array}{ccccc}
-1 & \le & \sin\dfrac{1}{x} & \le & 1, \\
\int_0^{|x|^{1+\alpha}} -1\ dt & \le & \int_0^{|x|^{1+\alpha}} \sin\frac{1}{t}\ dt & \le & \int_0^{|x|^{1+\alpha}} 1\ dt, \\
\dfrac{\int_0^{|x|^{1+\alpha}} -1\ dt}{|x|} & \le & \dfrac{\int_0^{|x|^{1+\alpha}} \sin\frac{1}{t}\ dt}{x} & \le & \dfrac{\int_0^{|x|^{1+\alpha}} 1\ dt}{|x|}, \\
\shortparallel & & & & \shortparallel \\
-\dfrac{|x|^{1+\alpha}}{|x|} = -|x|^\alpha & & & & \dfrac{|x|^{1+\alpha}}{|x|} = |x|^\alpha
\end{array}$$

$\lim_{x\to 0} -|x|^\alpha = \lim_{x\to 0} |x|^\alpha = 0$, by Squeeze Theorem, the limit is 0.

`Ans: A`

4.

$$\begin{aligned}
\int x^n e^{-x}\ dx &= x^n(-e^{-x}) - \int (-e^{-x}) n x^{n-1}\ dx \\
&= -x^n e^{-x} + \int n x^{n-1} e^{-x}\ dx \\
&= -(x^n + n x^{n-1}) e^{-x} + \int n(n-1) x^{n-2} e^{-x}\ dx = \cdots \\
&= -(x^n + n x^{n-1} + \cdots + n!x + n!) e^{-x},
\end{aligned}$$

$$\lim_{t\to\infty} t^n e^{-t} = \lim_{t\to\infty} \frac{t^n}{e^t} \overset{l'H}{=} 0.$$

$$\left(0\cdot\infty \to \frac{\infty}{\infty}, n \text{ times.}\right)$$

$$\begin{aligned}
\int_0^\infty x^n e^{-x}\ dx &= \lim_{t\to\infty} \int_0^t x^n e^{-x}\ dx \\
&= \lim_{t\to\infty} -(x^n + n x^{n-1} + \cdots + n!x + n!) e^{-x}\Big|_{x=0}^{t} \\
&= \lim_{t\to\infty} \left[n! - (t^n + n t^{n-1} + \cdots + n!t + n!) e^{-t}\right] = n!.
\end{aligned}$$

`Ans: C`

5.

$$\int_0^1 \left(e^{2x} - x\right)\ dx = \frac{e^{2x}}{2} - \frac{x^2}{2}\Big|_0^1 = \left(\frac{1}{2}e^2 - \frac{1}{2}\right) - \left(\frac{1}{2} - 0\right) = \frac{1}{2}e^2 - 1.$$

`Ans: B`

6.

(A) The average value of f on the interval $[1,4]$ is equal to

$$\frac{1}{4-1}\int_1^4 f(x)\,dx = \frac{10}{3} > 3.$$

(B) Let $f : [1, 4] \to \mathbb{R}$ such that $f(x) = \dfrac{20}{3}(x - 2)$.

Then

$$\int_1^4 f(x)\,dx = \frac{10}{3}(x-2)^2\Big|_{x=1}^{4} = 10,$$

and $\max_{x\in[1,4]} f(x) = f(4) = \dfrac{40}{3} > 4$.

(C) Let $f : [1, 4] \to \mathbb{R}$ such that $f(x) = \dfrac{20}{3}(x - 2)$.

Then

$$\int_1^4 f(x)\,dx = \frac{10}{3}(x-2)^2\Big|_{x=1}^{4} = 10,$$

and $\min_{x\in[1,4]} f(x) = f(1) = \dfrac{-20}{3} < 0$.

(D) If not, $\max_{x\in[1,4]} f(x) \le 3$, then

$$\int_1^4 f(x)\,dx \le \int_1^4 3\,dx = (4-1)\cdot 3 = 9 < 10,$$

it is a contradiction.

`Ans: D`

7. The surface area is

$$\begin{aligned}
S = \int 2\pi x\,dS &= \int_0^2 2\pi\left(e^t - t\right)\sqrt{(e^t-1)^2 + (2e^{t/2})^2}\,dt \\
&= \int_0^2 2\pi\left(e^t - t\right)\sqrt{(e^t+1)^2}\,dt \\
&= \int_0^2 2\pi\left(e^t - t\right)\left(e^t + 1\right)\,dt \\
&= 2\pi\int_0^2 e^{2t} - te^t + e^t - t\,dt \\
&= 2\pi\left(\frac{e^{2t}}{2} - te^t + 2e^t - \frac{t^2}{2}\Big|_0^2\right) \\
&= \pi\left(e^4 - 9\right).
\end{aligned}$$

`Ans: A`

8. Notice that $(x, y) = (-1, 1)$ if and only if $t = \frac{\pi}{2}$. The slope of the tangent line is

$$\frac{dy}{dx} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=\frac{\pi}{2}} = \left.\frac{\cos t + 2\cos 2t}{-\sin t - 2\sin 2t}\right|_{t=\frac{\pi}{2}} = \frac{-2}{-1} = 2.$$

Thus the tangent line is

$$y - y\left(\frac{\pi}{2}\right) = 2\left(x - x\left(\frac{\pi}{2}\right)\right),$$

that is, $y = 2x + 3$.

`Ans: A`

9. Let $r_1(\theta) = \frac{1}{2}$ and $r_2(\theta) = \cos 2\theta$. Notice that

$$\begin{aligned}
&(r_1(\theta_1)\cos\theta_1, r_1(\theta_1)\sin\theta_1) = (r_2(\theta_2)\cos\theta_2, r_2(\theta_2)\sin\theta_2)\\
\Leftrightarrow\ &\left(\theta_1 = \theta_2 \text{ and } \cos 2\theta_2 = \frac{1}{2}\right) \text{ or } \left(\theta_1 = \theta_2 + \pi \text{ and } \cos 2\theta_2 = -\frac{1}{2}\right)\\
\Leftrightarrow\ &\theta_2 = \frac{n\pi}{2} \pm \frac{\pi}{6} \quad \forall\, n \in \mathbb{Z}.
\end{aligned}$$

Notice that while $0 \le \theta \le \frac{\pi}{6}$, $0 \le r_1(\theta) = \frac{1}{2} \le r_2(\theta) = \cos 2\theta$. Hence the **area** A is equal to

$$\begin{aligned}
&8\int_0^{\pi/6} \frac{1}{2}\left[{r_2}^2(\theta) - {r_1}^2(\theta)\right]\, d\theta = 4\int_0^{\pi/6}\left[\cos^2(2\theta) - \frac{1}{4}\right]\, d\theta\\
= &4\int_0^{\pi/6}\left[\frac{1}{4} + \frac{1}{2}\cos(4\theta)\right]\, d\theta = \frac{\pi}{6} + \left(\left.\frac{1}{2}\sin(4\theta)\right|_{\theta=0}^{\pi/6}\right) = \frac{\pi}{6} + \frac{\sqrt{3}}{4}.
\end{aligned}$$

`Ans: B`

10. Let $y = (\ln x)^{\sin x}$. $\ln y = (\sin x)\ln(\ln x)$, and

$$\frac{y'}{y} = \cos x \cdot \ln(\ln x) + \sin x \cdot \frac{1}{\ln x} \cdot \frac{1}{x},$$

Thus

$$\begin{aligned}
f'(e) = y'|_{x=e} &= y\left.\left(\cos x \cdot \ln(\ln x) + \sin x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}\right)\right|_{x=e}\\
&= (\ln e)^{\sin e}\left(\cos e \cdot \ln(\ln e) + \sin e \cdot \frac{1}{\ln e} \cdot \frac{1}{e}\right)\\
&= 1 \cdot \left(\cos e \cdot 0 + \sin e \cdot \frac{1}{1} \cdot \frac{1}{e}\right) = \frac{\sin e}{e}.
\end{aligned}$$

`Ans: D`

11.

(A)

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x}{x} = \frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}.$$

(B) Since

$$\frac{-1}{|x|} \leq \frac{\sin x}{x} \leq \frac{1}{|x|}$$

for all $x \neq 0$, and

$$\lim_{x\to\infty} \frac{-1}{|x|} = 0 = \lim_{x\to\infty} \frac{1}{|x|}.$$

By squeeze theorem,

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

(C) Since $0 < \sin x < x < \tan x$ for $0 < x < \pi/2$, we have that

$$1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x} = \frac{1}{\cos x}.$$

Because $\lim_{x\to 0^+} \dfrac{1}{\cos x} = \dfrac{1}{1} = 1$, by squeeze theorem, we have that $\lim_{x\to 0^+} \dfrac{x}{\sin x} = 1$. For the similar way, we can prove that

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1.$$

(D) Since

$$\frac{-1}{|x|} \leq \frac{\sin x}{x} \leq \frac{1}{|x|}$$

for all $x \neq 0$, and

$$\lim_{x\to -\infty} \frac{-1}{|x|} = 0 = \lim_{x\to\infty} \frac{1}{|x|}.$$

By squeeze theorem,

$\lim_{x\to -\infty} \dfrac{\sin x}{x} = 0$, $\lim_{x\to -\infty} \dfrac{x - \sin x}{x} = 1 - 0 = 1$.

`Ans: CD`

12.

(A)

$$\lim_{x\to a} |f(x)| = \left|\lim_{x\to a} f(x)\right| = |f(a)|.$$

(B) Let $f:(0,\infty)\to\mathbb{R}$ such that$f(x)=1/x$ for all $x\in(0,\infty)$, and $g:[0,\infty)\to\mathbb{R}$ such that

$$g(x)=\begin{cases} f(x), & \text{if } x\in(0,\infty),\\ 0\ , & \text{if } x=0.\end{cases}$$

Then $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} g(x)=+\infty$, both of f and g have vertical asymptote $x=0$.

(C)

$$\lim_{x\to a} f(x)-f(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)=f'(a)\cdot(a-a)=0,$$

thus $\lim_{x\to a} f(x)=f(a)$.

(D) The limit of $\sin x$ does not exist as $x\to+\infty$ since $\sin n\pi=0$ and $\sin\left(\frac{\pi}{2}+2n\pi\right)=1$ for all $n\in\mathbb{N}$.

`Ans: ABC`

13.

$$\begin{aligned} f'(a)&=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=1.\\ \lim_{x\to a}\frac{f(x)-f(a)}{\sqrt[3]{x}-1}&=\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\,\frac{x-a}{\sqrt[3]{x}-1}\right)\\ &=\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}\frac{x-a}{\sqrt[3]{x}-1}\right)\\ &=f'(a)\left(\lim_{x\to a}\frac{x-a}{\sqrt[3]{x}-1}\right)\\ &=\begin{cases}\dfrac{a-a}{\sqrt[3]{a}-1}=0, & \text{if } a\neq 1,\\ \sqrt[3]{a^2}+\sqrt[3]{a}+1=3, & \text{if } a=1.\end{cases}\end{aligned}$$

`Ans: AD`

14.

(A)

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x\ln|x|=\lim_{x\to 0}\frac{\ln|x|}{1/x}\overset{l'H}{=}\lim_{x\to 0}\frac{1/x}{-1/x^2}=\lim_{x\to 0}(-x)=0=f(0),$$

f is continuous at 0.

(B) $f'(x)=\ln|x|+1=0$ if and only if $x=\pm e^{-1}$, and $f'(x)$ does not exist when $x=0$. For $x\neq 0$, $f''(x)=\dfrac{1}{x}$, $f''(e^{-1})=e>0$. Hence f has a local minimum $f(e^{-1})=-e^{-1}$.

(C) $f'(x)=\ln|x|+1=0$ if and only if $x=\pm e^{-1}$, and $f'(x)$ does not exist when $x=0$. For $x\neq 0$, $f''(x)=\dfrac{1}{x}$, $f''(-e^{-1})=-e>0$. Hence f has a local maximum $f(-e^{-1})=e^{-1}$.

(D)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\ln|x| = \lim_{x\to 0} \frac{\ln|x|}{1/x} \overset{l'H}{=} \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0 = f(0),$$

hence f is continuous at 0.

$f'(x) = \ln|x| + 1 = 0$ if and only if $x = \pm e^{-1}$, and $f'(x)$ does not exist when $x = 0$. For $x \neq 0$, $f''(x) = \frac{1}{x}$. Since $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, f is concave downward on $(-\infty, 0)$ and f is concave upward on $(0, \infty)$. In summary, f has an inflection point $(0, 0)$.

`Ans:` `ABCD`

15.

(A) $|x|$ is continuous,

$$\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)| = |f(a)|.$$

(B) Just follow the definition of vertical asymptote.

(C)

$$\lim_{x\to a} f(x) = f'(a)\cdot 0 + f(a) = f(a).$$

(D) $\lim_{x\to\infty} f(x)$ may not exist. (ex: $\sin x$.)

`Ans:` `ABD`

16. Implicit differentiate with respect to x,

$$\begin{aligned} & 3\left(x^2+y^2-1\right)^2(2x+2yy') = 3x^2+y' \\ \Rightarrow\ & 3\left(1^2+0^2-1\right)^2(2\cdot 1+2\cdot 0\cdot y') = 3\cdot 1^2+y' \\ \Rightarrow\ & y'\big|_{(x,y)=(1,0)} = -3. \end{aligned}$$

Thus the tangent line to the curve at $(0, 1)$ is $y - 0 = -3(x-1)$, that is, $y = -3x + 3$

`Ans:` $y = -3x + 3$

17.

$$\frac{9x^4+x^3+6x^2+5}{x^3+2x^2+x+5} = 9x - 17 + \frac{31x^2-28x+90}{x^3+2x^2+x+5},$$

and

$$\lim_{x\to\pm\infty} \frac{31x^2-28x+90}{x^3+2x^2+x+5} = 0.$$

So the slant asymptote of $f(x)$ is $y = 9x - 17$ or $9x - y - 17 = 0$.

`Ans:` $9x - y - 17 = 0$

18. Assume $\dfrac{(x-4)}{(x+1)(x^2+4)} = \dfrac{A}{x+1} + \dfrac{Bx}{x^2+4} + \dfrac{C}{x^2+4}$,
$x - 4 = A(x^2+4) + Bx(x+1) + C(x+1)$,

$$\begin{cases} x=-1: & -5 = 5A, \\ x = 0: & -4 = 4A + C, \\ x = 1: & -3 = 5A + 2B + 2C, \end{cases} \quad \Rightarrow \begin{cases} A = -1, \\ B = 1, \\ C = 0. \end{cases}$$

$$\int_0^1 \frac{(x-4)}{(x+1)(x^2+4)}\,dx = \int_0^1 \frac{-1}{x+1} + \frac{x}{x^2+4}\,dx$$
$$= -\ln|x+1| + \frac{1}{2}\ln(x^2+4)\Big|_{x=0}^{1} = (-\ln 2 + \ln\sqrt{5}) - (0 + \ln 2)$$
$$= \frac{1}{2}\ln 5 - 2\ln 2\ (= \ln\frac{\sqrt{5}}{4}).$$

`Ans:` $\dfrac{1}{2}\ln 5 - 2\ln 2\ (= \ln\dfrac{\sqrt{5}}{4})$

19. [Cylindrical shell]
(It is hard to use Disk method: $\int_0^1 \pi\left[f^{-1}(y)\right]^2\,dy$.)

$$\int_0^\pi 2\pi x f(x)\,dx = \int_0^\pi 2\pi x\frac{\sin x}{x}\,dx = \int_0^\pi 2\pi\sin x\,dx$$
$$= 2\pi\left(-\cos x|_{x=0}^{\pi}\right) = 2\pi[-(-1)-(-1)] = 4\pi.$$

`Ans:` 4π

20. $f(x) > \dfrac{1}{(x+1)[\ln(x+1)]^p} > 0$ for $x \geq 1$.

Now
$$\int_1^\infty \frac{1}{(x+1)[\ln(x+1)]^p}\,dx \overset{u=\ln(x+1)}{=} \int_{\ln 2}^\infty \frac{1}{u^p}\,du$$
diverges if and only if $p \leq 1$.
By comparison theorem, $\int_1^\infty f(x)\,dx$ diverges for $p \leq 1$.

For $x \geq e$, $\ln(x+1) > \ln x > 0$, $[\ln(x+1)]^p > (\ln x)^p$,

$0 < f(x) < \dfrac{1}{x[\ln x]^p}$.

Now,

$$\int_e^\infty \frac{1}{x[\ln x]^p}\,dx \overset{v=\ln x}{=} \int_1^\infty \frac{1}{v^p}\,dv$$

converges if and only if $p > 1$.

By comparison theorem, $\int_e^\infty f(x)\,dx$ converges when $p > 1$.
In summary, $\int_1^\infty f(x)\,dx$ converges for $p > 1$.

`Ans:` $p > 1$