Calculus 102-2

1. $r(t) =< 2t, t^2, \frac{t^2}{3} >$.
   $r'(t) =< 2, 2t, t^2 >$
   $r''(t) =< 0, 2, 2t >$
   $r'(1) \times r''(1) =< 2, -4, 4 >$
   $r(1) \cdot (r'(1) \times r''(1)) =< 2, 1, \frac{1}{3} > \cdot < 2, -4, 4 >= \frac{4}{3}$.

2. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(5x-4)^{n+1}}{(n+1)^3}}{\frac{(5x-4)^n}{n^3}} \right| = |5x-4| < 1. \Rightarrow \frac{3}{5} < x < 1.$
   Check the end points: $x = 1$ and $x = \frac{3}{5}$:
   (i) $x = 1$:
   $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.
   (ii) $x = \frac{3}{5}$:
   $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is convergent.

   The interval of convergence for $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$ is $[\frac{3}{5}, 1]$.

3. $f_x(0,0) = \lim_{h\to 0} \frac{0-0}{h-0} = 0$
   For $(x,y) \neq (0,0)$, we have $f_x(x,y) = y\frac{x^a - y^b}{x^2+y^2} + xy\frac{(ax^{a-1} - y^b)(x^2+y^2) - (x^a - y^b)(2x)}{(x^2+y^2)^2}$
   Then
   $$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h(-h^b)}{h^2} - 0}{h} = \lim_{h\to 0} \frac{-h^{b+1}}{h^3} = -1$$
   $\Rightarrow b = 2$.

4. $R = [0, \frac{1}{4}] \times [\frac{1}{4}, \frac{1}{2}]$.

$$\begin{aligned}
\int\int_R \sin \pi x \cos \pi y dA &= \int_0^{\frac{1}{4}} \sin \pi x dx \int_{\frac{1}{4}}^{\frac{1}{2}} \cos \pi y dy \\
&= \left(-\frac{1}{\pi} \cos \pi x \Big|_0^{\frac{1}{4}} \right) \left(\frac{1}{\pi} \sin \pi y \Big|_{\frac{1}{4}}^{\frac{1}{2}} \right) \\
&= -\frac{1}{\pi^2} \left(\frac{1}{\sqrt{2}} - 1 \right) \left(1 - \frac{1}{\sqrt{2}} \right) \\
&= \frac{1}{2\pi^2}(3 - 2\sqrt{2})
\end{aligned}$$

5. $r_1'(0) =< 3, 8, -4 >$, $r_2' =< 2, 6, 2 >$.
   $r_1'(0) \times r_2'(1) =< 40, -14, 2 >$
   $\Rightarrow S : 40x - 14y + 2z = 72 \Rightarrow z = -20x + 7y - 36$, then $f_x(2,1) = -20$.

6. $f(x,y) = x^4 - 4xy + 8y^2 - 1$.
   $\begin{cases} f_x(x,y) = 4x^3 - 4y = 0 \\ f_y(x,y) = -4x + 16y = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 \\ x = 4y \end{cases}$
   $\Rightarrow$ Critical points: $(0,0), (\frac{1}{2}, \frac{1}{8}), (-\frac{1}{2}, -\frac{1}{8})$.
   $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 16$, $f_{xy}(x,y) = f_{yx}(x,y) = -4$
   Then $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 16(12x^2 - 1)$.

(i) $(x, y) = (0, 0)$:
$D(0, 0) < 0$, then $(0, 0)$ is a saddle point.

(ii) $(x, y) = (\frac{1}{2}, \frac{1}{8})$:
$D(\frac{1}{2}, \frac{1}{8}) > 0$, $f_{xx}(\frac{1}{2}, \frac{1}{8}) > 0$, then f has a local minimum at $(\frac{1}{2}, \frac{1}{8})$.

(iii) $(x, y) = (-\frac{1}{2}, -\frac{1}{8})$:
$D(-\frac{1}{2}, -\frac{1}{8}) > 0$, $f_{xx}(-\frac{1}{2}, -\frac{1}{8}) > 0$, then f has a local minimum at $(-\frac{1}{2}, -\frac{1}{8})$.

7. Let $x = r\cos\theta$, $y = r\sin\theta$,

$$\begin{aligned}\int\int_R (3x + 4y^2)dA &= \int_0^{\pi}\int_1^3 \left(3r\cos\theta + 4r^2\sin^2\theta\right) r dr d\theta \\ &= \int_0^{\pi} r^3\cos\theta + r^4\sin^2\theta\Big|_1^3 d\theta \\ &= \int_0^{\pi} 26\cos\theta + 80\sin^2\theta d\theta \\ &= 26\sin\theta\Big|_0^{\pi} + \int_0^{\pi} 40(1 - \cos 2\theta)d\theta \\ &= 40\pi\end{aligned}$$

8. Let $\begin{cases} u = x + y \\ v = y + z \\ w = z + x \end{cases} \Rightarrow \dfrac{u + v + w}{2} = x + y + z \Rightarrow \begin{cases} x = \frac{u+v+w}{2} - v \\ y = \frac{u+v+w}{2} - w \\ z = \frac{u+v+w}{2} - u \end{cases}$, $\left|\dfrac{\partial(x, y, z)}{\partial(u, v, w)}\right| = \frac{1}{2}$

$\Rightarrow \displaystyle\int\int\int_{\{(x,y,z)|(x+y)^2+(y+z)^2+(z+x)^2\le 1\}} 1 dV = \int\int\int_{\{(u,v,w)|u^2+v^2+w^2\le 1\}} 1 \cdot \frac{1}{2} du dv dw = \frac{2}{3}\pi$

9. (A) Let $a_n = \dfrac{n^2 - 3}{n^3 + 5}$, $b_n = \frac{1}{n}$,
$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2-3}{n^3+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3 - 3n}{n^3 + 5} = 1.$
By limit comparsion test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 3}{n^3 + 5}$ is divergent.

(B) Let $a_n = \ln\frac{n}{3n+1}$, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\frac{n}{3n + 1} = \ln\frac{1}{3} \neq 0$. Then $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{3n + 1}$ is divergent.

(C) Since $0 \le \left|\frac{\sin n}{1+(1.5)^n}\right| \le \frac{1}{1+(1.5)^n} \le \frac{1}{(1.5)^n}$, then $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{1 + (1.5)^n}$ is convergent.

(D) Let $a_n = \frac{\sqrt{n}}{n+2}$, $b_n = \frac{1}{\sqrt{n}}$, $\displaystyle\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+2}}{\frac{1}{\sqrt{n}}} = 1$
By limit comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n + 2}$ is divergent.

10. Let $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. Then

$$\begin{cases} z=\sqrt{3x^2+3y^2} \\ x^2+y^2+z^2=z \end{cases} \Rightarrow \begin{cases} \rho\cos\phi=\sqrt{3}\rho\sin\phi \\ \rho^2\sin^2\phi+\rho^2\cos^2\phi=\rho\cos\phi \end{cases} \Rightarrow \begin{cases} \phi=\frac{\pi}{6} \\ \rho=0,\cos\phi \end{cases}$$

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{\cos\phi} 1\rho^2\sin\phi d\rho d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \frac{1}{3}\cos^3\phi\sin\phi d\phi d\theta \\ &= \int_0^{2\pi}\int_1^{\frac{\sqrt{3}}{2}} -\frac{1}{3}u^3 du d\theta \quad (u=\cos\phi) \\ &= \int_0^{2\pi}\int_{\frac{\sqrt{3}}{2}}^{1} \frac{1}{3}u^3 du d\theta \\ &= \frac{7}{96}\pi \end{aligned}$$

11. (A) Let $x=r\cos\theta$, $y=r\sin\theta$, $\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{r\to 0}\frac{r^2\cos\theta\sin\theta}{r^2}$ does not exist.
    $\Rightarrow f$ is not continous at $(0,0)$.
    (B) $f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h-0}=\lim_{h\to 0}\frac{0-0}{h}=0.$
    (C) By (A), f is not differentiable at $(0,0)$.
    (D) $\int\int_B f(x,y)dA=\int_0^{2\pi}\int_0^1\frac{r^2\cos\theta\sin\theta}{r^2}rdrd\theta=\int_0^{2\pi}\frac{1}{2}\cos\theta\sin\theta d\theta=\frac{1}{4}\int_0^{2\pi}\sin 2\theta d\theta=0$

12. (A) $\nabla f=\left(\frac{2xy^2}{x^2y^2},\frac{2x^2y}{x^2y^2}\right)$, $\nabla f(1,2)=(2,1)$
    (B) $\vec{u}=\frac{1}{\sqrt{5}}<2,1>$, $|\vec{u}|=1$. $D_uf(1,2)=(2,1)\cdot\frac{1}{\sqrt{5}}<2,1>=\sqrt{5}$
    (C) True
    (D) $\sqrt{5}$.

13. (A) False, counter example: $a_n=\frac{1}{n}$.
    (B) True.
    (C) False, counter example: $\{a_n\}=\{(-1)^n\}$, $\{b_n\}=\{(-1)^n\}$, $\{a_nb_n\}=\{1\}$.
    (D) True.

14. $D=\{(x,y)|x^2+y^2\le 1, x\le y\}$.
    $c=\int\int_D(x^2+y^2)dA=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\int_0^1 r^2rdrd\theta=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\int_0^1 r^3drd\theta.$
    Choose (A), (B), (D)

15. (A) Let $x=my^2$, $\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^4}=\lim_{y\to 0}\frac{my^4}{(m^2+1)y^4}=\frac{m}{m^2+1}$
    Thus $\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^4}$ doest not exist.
    (B) Since $0\le\left|\frac{3xy^2}{x^2+y^2}\right|\le 3\left|\frac{xy^2}{y^2}\right|\le 3|x|$.
    By Squeeze theorem, we have $\lim_{(x,y)\to(0,0)}\frac{3xy^2}{x^2+y^2}=0$
    (C) Let $x=r\cos\theta$, $y=r\sin\theta$, then
    $\lim_{(x,y)\to(0,0)}(x^2+y^2)\ln(x^2+y^2)=\lim_{r\to 0}r^2\ln r^2=\lim_{r\to 0}\frac{\ln r}{\frac{1}{r}}=0$(By L'Hospital's Rule).

(D) Let $x = r\cos\theta$, $y = r\sin\theta$, then $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0}\frac{\sin r^2}{r^2} = 1$

1. $w(x, y, z)$ satisfy $yw + xw + 3z^3 + zw - 6xz = 0$.

$$\begin{aligned} & y\frac{\partial w}{\partial z} + x\frac{\partial w}{\partial z} + 6z + w\frac{\partial w}{\partial z}z - 6x = 0 \\ \Rightarrow\ & \frac{\partial w}{\partial z}(y + x + z) = 6x - 6z - w \\ \Rightarrow\ & \left.\frac{\partial w}{\partial z}\right|_{(x,y,z)=(1,1,1)} = -\frac{1}{3} \end{aligned}$$

2. $f(x, y, z) = x^2 + y^2 + z^2$, $g(x) = x^3y^2z - 2$.

$$\begin{cases} \nabla f = \lambda\nabla g \\ x^3y^2z = 2 \end{cases} \Rightarrow \begin{cases} 2x = \lambda(3x^2y^2z) \\ 2y = \lambda(2x^3yz) \\ 2z = \lambda(x^3y^2) \\ x^3y^2z = 2 \end{cases} \Rightarrow \begin{cases} \lambda(3xy^2z) = 2 \\ \lambda(x^3y) = 1 \\ \lambda(x^3y^2) = 2z \\ x^3y^2z = 2 \end{cases}$$

Solve the (x, y, z, λ), the shortest distance between the points on the surface is $\sqrt[4]{12}$.

3. $r(t) =< e^{3t}\cos 4t, e^{3t}\sin 4t, e^3 >$, $0 \le t \le \ln 2$.

$$\begin{aligned} \text{Arc length of } r(t) &= \int_0^{\ln 2} |r'(t)|dt \\ &= \int_0^{\ln 2} \left|< 3e^{3t}\cos 4t + e^{3t}(-\sin 4t)\cdot 4, 3e^{3t}\sin 4t + 4e^{3t}\cos 4t, 0 >\right| dt \\ &= \int_0^{\ln 2} \sqrt{(3e^{3t}\cos 4t + e^{3t}(-\sin 4t)\cdot 4)^2 + (3e^{3t}\sin 4t + 4e^{3t}\cos 4t)^2}dt \\ &= \int_0^{\ln 2} 5e^{3t}dt = \frac{35}{3} \end{aligned}$$

4. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{4^{2(n+1)}(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{4(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1} = -\frac{1}{4}\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{8}$

5.

$$\begin{aligned} V &= 4\int_0^1\int_0^{\frac{1-x}{2}}\int_0^{\frac{1-x-2y}{3}} 1dzdydx = 4\int_0^1\int_0^{\frac{1-x}{2}}\frac{1-x-2y}{3}dydx \\ &= 4\int_0^1 \frac{1-x}{3}\frac{1-x}{2} - \frac{1}{3}\left(\frac{1-x}{2}\right)^2 dx = 4\int_0^1 \frac{(1-x)^2}{6} - \frac{1}{12}(1-x)^2dx \\ &= \frac{2}{9} \end{aligned}$$