Calculus 103-1

1.

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \left.\frac{6\sin^2 t\cos t}{6\cos^2 t(-\sin t)}\right|_{t=\frac{\pi}{4}} = -\left.\frac{\sin t}{\cos t}\right|_{t=\frac{\pi}{4}} = -1.$$

When $t = \frac{\pi}{4}$, $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then tangent: $y - \frac{1}{\sqrt{2}} = -(x - \frac{1}{\sqrt{2}})$.

2. (skip)

3. $f(x) = e^{\tan x}$, $f'(x) = e^{\tan x}\sec^2 x$.

4. $y = ax^3 + e^x$, $y'3ax^2 + e^x$, $y = 6ax + e^x$.
If $a = 0$, $y'' > 0$ for all x. Then $y = ax^3 + e^x$ has no inflection point.

5. (i) Vertical asymptotes: $x = \frac{5}{3}$.
   (ii) Horizontal aymptotes: $y = \frac{1}{\sqrt{3}}$, $y = -\frac{1}{\sqrt{3}}$.

6.

$$\begin{aligned} (\sin 2x)' &= 2\cos 2x \\ (\sin 2x)'' &= 2^2(-\sin 2x) \\ (\sin 2x)''' &= 2^3(-\cos 2x) \\ (\sin 2x)^{(4)} &= 2^4(\sin 2x) \\ &\vdots \\ (\sin 2x)^{(104)} &= 2^{104}\sin 2x. \end{aligned}$$

7. $F(x) = \int_3^x f(t)dt = -\int_x^3 f(t)dt$.

   (A) $F(0) = -2 + 1 = -1$.
   (B) $F(1) = -\frac{1}{2} + 1 = \frac{1}{2}$.
   (C) $F(2) = 1$.
   (D) $F(3) = 0$.

8.

$$f(x) = \begin{cases} x|x| & \text{if } -1 \le x \le 1 \\ x^2 & \text{if } x > 1 \\ x^3 & \text{if } x < -1 \end{cases} = \begin{cases} -x^2 & \text{if } 1 \le x \le 0 \\ x^2 & \text{if } x \ge 0 \\ x^3 & \text{if } x \le -1 \end{cases}$$

By definition of f, we know that f is differentiable on interval $(-\infty, -1)$, $(-1, 0)$ and $(0, \infty)$.
We only check the differentiabity of f at points $x = 0$ and -1.
By simple assumption, we find

$$\lim_{x\to 0^+}\frac{f(x) - f(0)}{x - 0} = 0 = \lim_{x\to 0^-}\frac{f(x)^f(0)}{x - 0}.$$

Thus f is differentiable at 0 and $f'(0) = 0$.

9. The volumn of S is equal to S': the solid obtained by rotating $R' = \{(x, y)|x^2 + (y+1)^2 \le 1\}$ about the x-axis. Therefore, the volume of S is $2\pi^2$.

10.

$$\begin{aligned}
\text{Surface Area} &= \int_0^{\ln 2} 2\pi \frac{e^y + e^{-y}}{2} \sqrt{1 + \left(\frac{e^y + e^{-y}}{2}\right)^2}\, dy \\
&= \int_0^{\ln 2} \pi(e^y + e^{-y}) \frac{e^y + e^{-y}}{2} dy \\
&= \frac{\pi}{2} \int_0^{\ln 2} e^{2y} + 2 + e^{-2y} dy \\
&= \frac{\pi}{2} \left(\frac{1}{2}e^{2y} + 2y - \frac{1}{2}e^{-2y} \Big|_0^{\ln 2} \right) \\
&= \pi(\frac{15}{16} + \ln 2).
\end{aligned}$$

11. (skip, by definition)

12. Since

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x-1}{x-1} = 1$$
$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{2-x-1}{x-1} = -1$$

Then $f'(1)$ does not exist.

13. By definition, choose (A), (D).

14. $\int_0^1 \pi(f^2(x) - 0^2)dx = \int_0^1 \pi f^2(x)dx = \int_0^1 2\pi y(1 - g(y))dy$.

15. Notice that $\frac{x^a}{1+x^2}$ is nonnegative on $[0, \infty)$. Now, we divided into two caes: $a \geq 0$ and $a < 0$.

(i) $a \geq 0$:
Since $1 + x^2 \leq 2x^2$ on $[1, \infty)$, we have

$$0 \leq \frac{1}{2x^{2-a}} \leq \frac{x^a}{1+x^2} \text{ on } [1, \infty)$$

Using p-series test, $\int_1^\infty \frac{1}{2x^{2-a}} dx$ will diverge if $2 - a \leq 1$. Then by comparision test, we know that $\int_0^\infty \frac{x^a}{1+x^2} dx$ will diverge if $1 \leq a$. Thus we can't choose (D).
On the other hand, it is easy to see that $\int_0^1 \frac{x^a}{1+x^2} dx \leq 1$ and

$$0 \leq \frac{x^a}{1+x^a} \leq \frac{1}{x^{2-a}} \text{ on } [1, \infty)$$

Using p-series Test, $\int_1^\infty \frac{1}{x^{2-a}} dx$ will converge if $2 - a > 1$. By comparsion test, $\int_0^\infty \frac{x^a}{1+x^2}$ will converge if $1 > a \geq 0$. Thus we choose (C).

(ii) $a < 0$:
Since $1 + x^2 \leq 2$ on $[0, 1]$, then $0 \leq \frac{x^a}{2} \leq \frac{x^a}{1+x^2}$ holds for $x \in [0, 1]$. By p-series test, $\int_0^\infty \frac{x^a}{1+x^2} dx$ will diverge if $a \leq -1$. Thus, we can't choose (A).
On the other hand, we know that $0 \leq \frac{x^a}{1+x^2} \leq \frac{1}{x^{2-a}}$ holds for $x \in [1, \infty)$. Then $\int_1^\infty \frac{1}{x^{2-a}} dx$ always converge when $a > 0$. By comparision test, $\int_1^\infty \frac{x^a}{1+x^2} dx$ will converge if $a < 0$. The key observaiton will give $0 \leq \frac{x^a}{1+x^2} \leq x^a$ holds for $x \in [0, 1]$. By p-series test, $\int_0^1 x^a dx$ converges if $-1 < a < 0$. Thus by comparision test, $\int_0^1 \frac{x^a}{1+x^2} dx$ will converge if $a > -1$.
From above, $\int_0^\infty \frac{x^a}{1+x^2} dx$ will converge if $-1 < a < 0$, choose (B).

1. Since $e^x >> x^{2015}(\ln x)^{14}$ when $x \to \infty$, then

$$\lim_{x\to\infty} \frac{x^{2015}(\ln x)^{14}}{e^x} = 0.$$

2.

$$\lim_{x\to\infty} \left(\frac{x+a}{x-a}^x\right) = e^{\lim_{x\to\infty}\left(\frac{\ln\frac{x+a}{x-a}}{\frac{1}{x}}\right)} = e^1$$

$$\Rightarrow \quad \lim_{x\to\infty} \left(\frac{\ln\frac{x+a}{x-a}}{\frac{1}{x}}\right) = \lim_{x\to\infty} \frac{\frac{x+a}{x-a}\,\frac{x-a-x-a}{(x-a)^2}}{(-1)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2ax^2}{(x+a)(x-a)} = 2a = 1$$

$$\Rightarrow \quad a = \frac{1}{2}.$$

3.

$$\begin{aligned}
\int_0^{e-2} \ln(2+x)dx &= \int_2^e \ln u du \text{ (Let } u = 2+x) \\
&= u \ln u \Big|_2^e - \int_2^e 1 du \\
&= e - 2\ln 2 - e + 2 \\
&= 2 - 2\ln 2.
\end{aligned}$$

4.

$$\begin{aligned}
\int_0^1 \sqrt{1+(y')^2}dx &= \int_0^1 \sqrt{1+(\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}})^2}dx \\
&= \int_0^1 \frac{1}{2}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})dx \\
&= \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\Big|_0^1\right] = \frac{4}{3}.
\end{aligned}$$

5.

$$\begin{aligned}
\int \frac{1}{x^8 - x}dx &= \int \frac{1}{x(x^7-1)}dx \\
&= \int \frac{x^6}{x^7(x^7-1)}dx \\
&\text{Let } u = x^7,\ du = 7x^6dx \\
&= \frac{1}{7}\int \frac{1}{u(u-1)}du \\
&= \frac{1}{7}\int -\frac{1}{u} + \frac{1}{u-1}du \\
&= \frac{1}{7}\left(-\ln|u| + \ln|u-1|\right) + C \\
&= \frac{1}{7}\ln\left|\frac{x^7-1}{x^7}\right| + C
\end{aligned}$$