Calculus 103-2

1. $f(x) = e^{x^3}$

   Since $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

   Then $e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$.

   Compare the coefficient of $x^{3n}$,

   $\Rightarrow \frac{f^{(3n)}(0)}{(3n)!} = \frac{1}{n!} \Rightarrow f^{(3n)}(0) = \frac{(3n)!}{n!}$.

2. Let $\{a_n\} = \left\{\sqrt{3}, \sqrt{3\sqrt{3}}, \cdots, \right\}$, $\Rightarrow a_{n+1} = \sqrt{3a_n}$.

   Let $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_{n+1} = L = \lim_{n\to\infty} \sqrt{3a_n} = \sqrt{3L} \Rightarrow L = 0$ or $L = 3$.

   Since $a_n$ is increasing and $a_1 = \sqrt{3} > 0$, $\lim_{n\to\infty} a_n = 3$.

3.

$$\begin{aligned} D_u f(0,0) &= \lim_{h\to 0} \frac{f(0+\frac{h}{\sqrt{2}}, 0+\frac{h}{\sqrt{2}}) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{(\frac{h}{\sqrt{2}})(\frac{h}{\sqrt{2}})^2}{(\frac{h}{\sqrt{2}})^2+(\frac{h}{\sqrt{2}})^2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{\frac{h}{\sqrt{2}}}{1+1}}{h} \\ &= \lim_{h\to 0} \frac{h}{2\sqrt{2}h} = \frac{\sqrt{2}}{4} \end{aligned}$$

4. $f(x,y,z) = \cos(xyz) - x^2y^2 - z$,

   $\nabla f = (-\sin(xyz)yz - 2xy^2, -\sin(xyz)xz - 2x^2y, -\sin(xyz)xy - 1)$

   $\nabla f(1,-1,0) = (-2, 2, -1)$

   tangent plane: $-2(x-1) + 2(y+1) - 1(z-0) = 0 \Rightarrow z = -2x + 2y + 4$

5. $f(x,y) = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + 4xy - y^2$.

   $\begin{cases} f_x(x,y) = -x^3 + 2x^2 + 4y = 0 \\ f_y(x,y) = 4x - 2y = 0 \end{cases} \Rightarrow \begin{cases} -x^3 + 2x^2 + 4y = 0 \\ y = 2x \end{cases}$

   $\Rightarrow x = 0, -2, 4$.

   Critical points: $(0,0), (-2,-4), (4,8)$.

   $f_{xx}(x,y) = -3x^2 + 4x$, $f_{yy}(x,y) = -2$, $f_{xy}(x,y) = f_{yx}(x,y) = 4$

   (i) $(x,y) = (0,0)$:
   $\because D(0,0) < 0, \Rightarrow (0,0)$ is a saddle point.

   (ii) $(x,y) = (-2,-4)$:
   $\because D(-2,-4) > 0, f_{xx}(-2,-4) < 0, \Rightarrow f$ has a local maximum at $(-2,-4)$

   (iii) $(x,y) = (4,8)$ :
   $\because D(4,8) > 0, f_{xx}(4,8) < 0, \Rightarrow f$ has a local maximum at $(4,8)$

6. If $\int\int\int_E (1 - x^2 - 2y^2 - 3z^2)dV$ has maximum value,

   $\Rightarrow 1 - x^2 - 2y^2 - 3z^2 \geq 0$ in $E$

   $\Rightarrow E : x^2 + 2y^2 + 3z^2 \leq 1$, choose (B).

7. $\int_0^1 \int_y^1 (1+x^2)^{-1} dxdy$
$= \int_0^1 \int_0^x \frac{1}{1+x^2} dydx$
$= \int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \ln|1+x^2|\Big|_0^1 = \frac{1}{2}\ln 2.$

8. Let $\Omega = \{(x,y) | (x+y)^2 + |x-y| \le 1\}$.
$\left\{ \begin{array}{l} u = x+y \\ v = x-y \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{array} \right., |J| = \frac{1}{2}, \Omega' = \{(u,v) | u^2 + |v| \le 1\}.$

$$\begin{aligned} \int\int_\Omega 1 dxdy &= \int\int_{\Omega'} 1\frac{1}{2} dvdu \\ &= \frac{1}{2}\left(\int\int_{u^2+v\le 1, v\ge 0} 1dvdu + \int\int_{u^2-v\le 1, v\le 0} 1dvdu\right) \\ &= \frac{1}{2}\left(2\int_{-1}^{1}\int_0^{1-u^2} 1dvdu\right) \\ &= \frac{4}{3}. \end{aligned}$$

9.

$$\begin{aligned} &\int\int\int_E \sin\left[(x^2+y^2+z^2)^{\frac{3}{2}}\right] dV \\ =& \int_0^{2\pi}\int_0^{\pi}\int_1^2 \sin(\rho^3)\rho^2 \sin\phi d\rho d\phi d\theta \\ =& \int_0^{2\pi}\int_0^{\pi} \sin\phi \left(-\frac{1}{3}\cos\rho^3\Big|_1^2\right) d\phi d\theta \\ =& \int_0^{2\pi}\int_0^{\pi} \sin\phi(-\frac{1}{3}\cos 8 + \frac{1}{3}\cos 1) d\phi d\theta \\ =& \frac{4}{3}\pi(\cos 1 - \cos 8). \end{aligned}$$

10.

$$\begin{aligned} &\int\int_D e^{x+y} dA \\ =& \int_0^1\int_0^{1-y} e^x e^y dxdy \\ =& \int_0^1 e^y \left[e^x\Big|_0^{1-y}\right] dy \\ =& \int_0^1 e - e^y dy \\ =& ey - e^y\Big|_0^1 = e - e + 1 = 1 \end{aligned}$$

11. (A) False
(B) True
(C) $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt = 3\sqrt{2}(b-a).$

(D) $\int_0^{\pi} (\cos 3t, \sin 3t, 3t)\,dt = (0, \frac{2}{3}, \frac{3}{2}\pi^2)$

12. (A) Let $a_n = \frac{\sin 4n}{4^n}$, $|a_n| = |\frac{\sin 4n}{4^n}| \le |\frac{1}{4^n}|$.
Since $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is convergent, then $\sum_{n=1}^{\infty} |a_n|$ is convergent.
$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is convergent.

(B) Let $a_n = \frac{1}{n+n\cos^2 n}$, $a_n \ge \frac{1}{n+n} = \frac{1}{2n}$.
Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent, then $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ is divergent.

(C) $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n-1}{n}\right) = \sum_{n=2}^{\infty} \ln(n-1) - \ln n$
$= \ln 1 - \ln 2 + \ln 2 - \ln 3 + \cdots = 0 - \lim_{n\to\infty} \ln n$ is divergent.

(D) Let $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \le \frac{1}{n(\sqrt{n}+\sqrt{n})} = \frac{1}{2n^{\frac{3}{2}}}$.
Since $\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{3}{2}}}$ is convergent, then $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ is convergent.

13. (A) False, counterexample: $a_n = -n$.

(B) True, $\left(\text{Since } a_n, b_n \ge 0, \quad \sum_{n=1}^{\infty} a_n b_n \le \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n\right)$

(C) $\lim_{n\to\infty} n^2 a_n = \lim_{n\to\infty} \frac{a_n}{\frac{1}{n^2}} = 0$ and $a_n > 0$
$\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

(D) False, counter example: $a_n = \frac{1}{n}$

14. (A) True, dy definition.

(B) False

(C) $\int f_x(x,y)dx = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 + g(y)$
$\int f_y(x,y)dy = \frac{1}{2}x^2 + y^2 + \frac{2}{3}y^3 + h(x)$
$\Rightarrow$ Let $f(x,y) = \frac{1}{3}x^3 + \frac{1}{2}x^2y^2 + \frac{2}{3}y^3$, then $f_x(x,y) = x^2 + xy^2$, $f_y(x,y) = x^2y + 2y^2$

(D) False

15. Let $a_n = (b^2)^{\ln n} = n^{\ln b^2} = \frac{1}{n^{\ln b^{-2}}}$.
If $\sum a_n$ is convergent, then $\ln b^{-2} > 1 \Rightarrow b^{-2} > e \Rightarrow -\frac{1}{e^2} < b < \frac{1}{e^2}$.

1. Area $= \int_0^{2\pi} \frac{1}{2}\left((\sqrt{\frac{\theta}{2\pi}})^2 - ((\frac{\theta}{2\pi})^2)^2\right) d\theta = \frac{1}{2}\int_0^{2\pi} \frac{\theta}{2\pi} - (\frac{\theta}{2\pi})^4 d\theta.\ (*)$
Let $u = \frac{\theta}{2\pi}$, $du = \frac{1}{2\pi}d\theta$,
$\Rightarrow (*) = \frac{1}{2}\int_0^1 (u - u^4)(2\pi)du = \frac{3}{10}\pi$

2. $x = r\cos\theta$, $y = r\sin\theta$, $\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = r$

3. $f(x,y) = \sqrt{x^2+y^2}\cos(yxe^{(x^2+y^2)^{\frac{3}{2}}})$.
   $f_x(-1,0) = \lim_{h\to 0} \dfrac{f(-1+h,0)-f(-1,0)}{h} = -1$

4. $\int\int_R xydA = \int_0^1 \int_{-x}^{x} xydydx + \int_1^2 \int_{x-2}^{2-x} xydydx = 0$

5. $\int\int\int 1dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 1rdzdrd\theta = \dfrac{\pi}{2}$.