Calculus 104-1

1. Since f has local extrema at $x=-2,1$, so $x=-2,1$ are critical points of f. Then $f'(-2)=f'(1)=0$. $\Rightarrow a=\frac{3}{2}, b=-6,\ a+b=-\frac{9}{2}$

2. Let $y(t)=t-t^2=0 \Rightarrow t=0$ or 1.

$$Area=\int_0^1 (t-t^2)e^t dt = 3-e.$$

3.

$$\int \sin x \cos x dx = \int \sin x\ d(\cos x) = \frac{1}{2}\sin^2 x + C.$$

4.

$$\begin{aligned} L &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2} dt \\ &= \int_0^1 \sqrt{\left(e^t \cos t + e^t(-\sin t)\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2} dt \\ &= \int_0^1 \sqrt{e^{2t}\left(1-2\cos t \sin t + 1 + 2\cos t \sin t\right)} dt \\ &= \int_0^1 \sqrt{2} e^t dt = \sqrt{2}(e-1). \end{aligned}$$

5.

$$V = 2\int_0^1 \pi r^2 dy = 2\int_0^1 \pi(1+y^2)dy = \frac{56\pi}{15}$$

6.

$$\lim_{x\to 0}\frac{x\sin x}{1-\cos 2x} = \lim_{x\to 0}\frac{\sin x + x\cos 2x}{2\sin 2x} = \lim_{x\to 0}\frac{\cos x + \cos x + x(-\sin x)}{4\cos 2x} = \frac{1}{2}.$$

7.

$$\begin{aligned} & F^{-1}(F(x)) = x \\ \Rightarrow\ & (F^{-1})'(F(x)) \cdot F'(x) = 1 \\ \Rightarrow\ & (F^{-1})'(F(x)) = \frac{1}{-3x^2\sqrt{2^{x^3}+14}} \\ \Rightarrow\ & (F^{-1})'(0) = -\frac{1}{12}.\ \text{(Since } F(1)=0\text{)} \end{aligned}$$

8.

$$\begin{aligned} \text{Surface Area} &= \int_0^1 2\pi x^3 \sqrt{1+\left(3x^2\right)^2} dx \\ &= \int_0^1 2\pi x^3 \sqrt{1+9x^4} dx \\ & \quad (\text{Let } u = 1+9x^4, du = 36x^3 dx) \\ &= \int_1^{10} 2\pi \frac{1}{36}\sqrt{u} du \\ &= \frac{1}{27}\left(10^{\frac{3}{2}}-1\right). \end{aligned}$$

9. Since $\sin x + 2\cos x = \sqrt{5}(\frac{1}{\sqrt{5}}\sin x + \frac{2}{\sqrt{5}}\cos x) = \sqrt{5}\sin(x+\alpha)$ for some α. Thus there exist $x_0 \in \mathbb{R}$ such that $\sin x_0 + 2\cos x_0$ attains its maximum $\sqrt{5}$.
On the other hand, the simple observaiton will gives g is a nonnegative, strictly increasing function on $\{x|x \geq 0\}$ and g is always negative on $\{x|x < 0\}$.
Therefore, the absolute maximum value of $g(\sin x + 2\cos x)$ must occurs when the function $\sin x + 2\cos x$ attain its maximumm. i.e. $\max_{x\in\mathbb{R}}\{g(\sin x + 2\cos x)\} = g(\sqrt{5}) = \sqrt{5}e^{\sqrt{5}}$.

10.
$$\begin{aligned}
& \lim_{n\to\infty}\left(\frac{1}{n^3\sqrt{4n^2+1}} + \frac{8}{n^3\sqrt{4n^2+4}} + \cdots + \frac{n^3}{n^3\sqrt{4n^2+n^2}}\right) \\
= & \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{4+\frac{1}{n^2}}}\left(\frac{1}{n^3}\right) + \frac{1}{\sqrt{4+\frac{4}{n^2}}}\left(\frac{8}{n^3}\right) + \cdots + \frac{1}{\sqrt{4+\frac{n^2}{n^2}}}\left(\frac{n^3}{n^3}\right)\right) \\
= & \int_0^1 \frac{x^3}{\sqrt{4+x^2}}dx \\
& (\text{Let } x = 2\tan\theta,\ dx = 2\sec\theta d\theta) \\
= & \int_0^{\tan^{-1}\frac{1}{2}} \frac{8\tan^3\theta}{2\sec\theta}2\sec^2\theta d\theta \\
= & \int_0^{\tan^{-1}\frac{1}{2}} 8\tan^3\theta\sec\theta d\theta \\
& (\text{Let } u = \sec\theta,\ du = \sec\theta\tan\theta) \\
= & 8\int_1^{\frac{\sqrt{5}}{2}}(u^2-1)du \\
= & -\frac{7\sqrt{5}}{3} + \frac{16}{3}.
\end{aligned}$$

11. Consider $a \geq 0$. Notice that $\frac{\sqrt{x}}{1+x^a}$ always nonnegative on $[0,\infty]$.
Since
$$0 \leq \frac{1}{2}\frac{1}{x^{a-\frac{1}{2}}} \leq \frac{\sqrt{x}}{1+x^2} \text{ on } [1,\infty]$$
Using p-series test, we know that if $a - \frac{1}{2} \leq 1$, $\int_1^\infty \frac{1}{x^{a-\frac{1}{2}}}dx$ will diverge. Thus from the comparision test, $\int_0^\infty \frac{\sqrt{x}}{1+x^a}dx$ also diverge. We can's choose (A) and (B).
On the other hand, the relation $\int_0^1 \frac{\sqrt{x}}{1+x^a}dx \leq 1$ always holds and the following relation
$$0 \leq \frac{\sqrt{x}}{1+x^a} \leq \frac{1}{x^{a-\frac{1}{2}}}$$
holds on $[1,\infty]$. Using p-series test again, we know if $a - \frac{1}{2} > 1$, $\int_1^\infty \frac{1}{x^{a-\frac{1}{2}}}dx$ will converge. Thus from the comparison test, $\int_1^\infty \frac{\sqrt{x}}{1+x^a}dx$ also converge. Therefore, we choose (C) and (D).

12. (i) Vertical asymptote: $x = 0$.
(ii) Let $y = ax + b$ is a slant asymptote.
(1) $x \to \infty$:
$$\begin{aligned}
a &= \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{\sqrt{e^{2x}+x^2}}{x(e^x-1)} = 0 \\
b &= \lim_{x\to\infty} f(x) - ax = \lim_{x\to\infty}\frac{\sqrt{e^{2x}+x^2}}{e^x-1} - 0\cdot x = 1.
\end{aligned}$$

(2) $x \to -\infty$:

$$\begin{aligned} a &= \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{e^{2x}+x^2}}{x(e^x-1)} = 1 \\ b &= \lim_{x\to-\infty} f(x) - ax = 0. \end{aligned}$$

Then $y = x$ and $y = 1$ are asymptotes.

13. (A) $\lim_{x\to 0} \dfrac{\tan x - x}{x} = \lim_{x\to 0} \frac{\sec^2 x - 1}{1} = 0.$

(B)

$$\begin{aligned} & 0 \le |x^3 \sin \frac{1}{x}| \le x^3 \\ \Rightarrow & \lim_{x\to 0} 0 \le \lim_{x\to 0} |x^3 \sin \frac{1}{x}| \le \lim_{x\to 0} x^3 = 0 \end{aligned}$$

By squeeze theorem,

$$\lim_{x\to 0} |x^3 \sin \frac{1}{x}| = 0 \Rightarrow \lim_{x\to 0} x^3 \sin \frac{1}{x} = 0.$$

(C) $\lim_{x\to 0^+} (1+x)^{1/x} = e.$

(D) $\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{\cos(x-1)}{2x+1} = \frac{1}{3}$

14. (A) True. (Since $x = 7$, f is not well-defined)

(B) $f(-x) = \frac{-2x+5}{7-x} \neq \frac{-2x-5}{x-7} = -f(x)$. Then $f(x)$ is not an odd function.

(C) $y = \frac{2x+5}{x-7} \Rightarrow xy - 7x = 2x + 5 \Rightarrow x = \frac{5+7y}{y-2}$
$\Rightarrow f^{-1}(x) = \frac{5+7x}{x-2}$.

(D) True. (Since $2x+5$, $x-7$ is continuous on $(-\infty, 7) \cup (7, \infty)$)

15. (A) $\int_{-\infty}^{\infty} x^3 dx = \int_{-\infty}^{0} x^3 dx + \int_0^\infty x^3 dx \Rightarrow$ divergent.

(B) True, since $\csc 1 > 1$, then by p-series test, $\int_3^\infty \frac{1}{x^{\csc 1}} dx$ is convergent.

(C) Let $f(x) = \frac{\tan x^3}{1+x^2}$, $f(-x) = \frac{-\tan x^3}{1+x^2} = -f(x)$.
$\Rightarrow f(x)$ is odd function
$\Rightarrow \int_{-\pi/4}^{\pi/4} f(x)dx = 0.$

(D) $\int_0^{\pi/4} \tan^2 x dx = \int_0^{\pi/4} \sec^2 x - 1 dx = \tan x - x|_0^{\pi/4} = 1 - \frac{\pi}{4}.$

1.

$$\begin{aligned} & x^3 + xy - \cos(xy) = 0 \\ \Rightarrow\ & 3x^2 + y + x\frac{dy}{dx} + \sin(xy)[y + x\frac{dy}{dx}] = 0 \\ & \text{Let } (x,y) = (1,0) \\ \Rightarrow\ & 3 + \frac{dy}{dx} = 0, \Rightarrow \frac{dy}{dx} = -3. \end{aligned}$$

2.

$$\begin{aligned} \int_0^1 (1-x)f(x^2-2x+1)dx &= \int_0^1 (1-x)f((x-1)^2)dx \\ & \quad \text{Let } u = (x-1)^2,\ du = 2(x-1)dx \\ &= \int_1^0 -\frac{1}{2} f(u)du \\ &= \int_0^1 \frac{1}{2} f(u)du = 2. \end{aligned}$$

3. $f(x) = \int_x^{x^3} \sin(t^2)dt$,
   $f'(x) = \sin((x^3)^2)3x^2 - \sin x^2 = 3x^2 \sin x^6 - \sin x^2$.

4. (i) tangent point: $(x(1), y(1)) = (e, 1)$

   (ii)
$$\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=1} = \left.\frac{1 - \frac{1}{t^2(2t)}}{e^{\sqrt{t}}\frac{1}{2\sqrt{t}}}\right|_{t=1} = -\frac{2}{e}.$$

   Then tangent: $y - 1 = -\frac{2}{e}(x - e)$.

5. Since f is differentiable on $[1, 4]$, by Mean-Value Theorem, there exist $\eta_1 \in (1, 2)$ and $\eta_2 \in (2, 4)$ such that
$$\begin{aligned} f(2) - f(1) &= f'(\eta_1)(2 - 1) = f'(\eta_1) \\ f(4) - f(2) &= f'(\eta_2)(4 - 2) = 2f'(\eta_2) \end{aligned}$$
   From assumption $f'(x) \geq 3$ for all $x \in [1, 4]$, then
$$\begin{aligned} f(2) - f(1) = f'(\eta_1) \geq 3 \\ f(4) - f(2) = f'(\eta_2) \geq 3 \end{aligned}$$
   Therefore,
$$4 = 3 + f(x) \leq f(2) \leq f(4) - 6 = 4$$
   $\Rightarrow f(2) = 4$.