Calculus 104-2

1. Let $x = r\cos\theta$, $y = r\sin\theta$, then

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\ &= \frac{\cos\theta\sin\theta + (2+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (2+\sin\theta)\sin\theta} \\ \frac{dy}{dx}|_{\theta=\frac{\pi}{4}} &= -1 - \frac{1}{\sqrt{2}}\end{aligned}$$

2. Area $= 4\int_0^1 y dx = 4\int_0^1 (1-x^{1/3})^3 dx = \frac{1}{5}$.

3. Let $a_n = \left[\ln(1+\frac{2}{\sqrt{n}})\right] x^n$.

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ =& \lim_{n\to\infty}\left|\frac{\ln(1+\frac{2}{\sqrt{n+1}})}{\ln(1+\frac{2}{\sqrt{n}})}x\right| \\ =& \lim_{n\to\infty}\frac{\frac{1}{1+\frac{2}{\sqrt{n+1}}}2(-\frac{1}{2})(n+1)^{-3/2}}{\frac{1}{1+\frac{2}{\sqrt{n}}}2(-\frac{1}{2})n^{-3/2}}|x| \\ =& |x| < 1\end{aligned}$$

(i) $x = 1$, $a_n = \ln(1+\frac{2}{\sqrt{n}})$, $\sum a_n$ is divergent

(ii) $x = -1$, $a_n = (-1)^n\ln(1+\frac{2}{\sqrt{n}})$, $\sum a_n$ is convergent

$\Rightarrow$ the interval of convergence of $\sum_{n=1}^{\infty}\left[\ln(1+\frac{2}{\sqrt{n}})\right]x^n$ is $[-1,1)$.

4. $f(x) = \int_0^x h(t)dt$,
$f'(x) = h(x) = \frac{\sin x}{x} = \frac{1}{x}(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$
$f^{(5)} = \frac{d^4(f'(x))}{dx^4} = \frac{1}{5}$

5. $y\frac{\partial F(x,y)}{\partial x} + x\frac{\partial F(x,y)}{\partial y} = y(f'(x^2-y^2)2x) + x(1+f'(x^2-y^2))(-2y) = x.$

6. Since $\sin y \approx y$ when $y \to 0$, then

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x\sin y}{x^2+y^2} &= \lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2} \\ &= \lim_{r\to 0}\frac{r^2\cos\theta\sin\theta}{r^2} \\ &= \text{doesn't exist.}\end{aligned}$$

7.

$$\begin{aligned}
& \int_0^1 \int_x^1 2\sin(y^2)dydx \\
= & \int_0^1 \int_0^y 2\sin(y^2)dxdy \\
= & \int_0^1 2y\sin(y^2)dy \\
= & -\cos(y^2)\Big|_0^1 = 1-\cos 1
\end{aligned}$$

8. $z = f(x,y) = x^2 + y^2$, $R = \{x^2 + y^2 = 2\}$.
Surface Area $= \int\int_R \sqrt{(f_x(x,y))^2 + f_y(x,y)^2 + 1}dxdy = \int\int_R \sqrt{4x^2 4y^2 + 1}dxdy$
$= \int_0^{2\pi}\int_0^{\sqrt{2}} (4r^2+1)^{\frac{1}{2}} r dr d\theta = \frac{13}{3}\pi.$

9. $a_0 = 0, a_2 = 1, a_3 = \frac{2}{3}, a_4 = \frac{7}{9}$.
From $a_{n+1} = \frac{2}{3}a_{n+1} + \frac{1}{3}a_n$, $a_n \geq \frac{2}{3}$, choose (D).

10. Let $\begin{cases} u = x+y \\ v = x-y \end{cases} \Rightarrow \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$, $|J| = |\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}| = \frac{1}{2}$

$$\begin{aligned}
\int\int_D \frac{x-y}{x+y}dA &= \frac{1}{2}\int_{-2}^0 \int_2^4 \frac{v}{u}dudv \\
&= \frac{1}{2}\int_{-2}^0 v\ln|u|\Big|_2^4 dv \\
&= \frac{1}{2}\ln 2\left(\frac{1}{2}v^2\Big|_{-2}^0\right) = -\ln 2.
\end{aligned}$$

11. $f(x,y) = -x^3 + 4xy - y^2 + 1$.
$\begin{cases} f_x(x,y) = -3x^2 + 4y = 0 \\ f_y(x,y) = 4x - 2y = 0 \end{cases}$
Then the critical points are $(0,0)$, $(\frac{8}{3}, \frac{16}{3})$.
$f_{xx}(x,y) = -6x,\ f_{yy}(x,y) = -2,\ f_{xy}(x,y) = f_{yx}(x,y) = 4.$

(i) $(x,y) = (0,0)$:
$D = f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = -16 < 0$
$\Rightarrow (0,0)$ is a saddle point.

(ii) $(x,y) = (\frac{8}{3}, \frac{16}{3})$: $D = 8 > 0$ and $f_{xx}(\frac{8}{3}, \frac{16}{3}) < 0$.
$\Rightarrow f$ has a local maximum at $(\frac{8}{3}, \frac{16}{3})$.

12. (A) False, counter example: $b_n = \frac{1}{n^2}$

(B) True, $\frac{a_n+b_n}{2} \geq \sqrt{a_n b_n}$
$\because \frac{1}{2}\sum(a_n + b_n)$ is convergent, $\Rightarrow \sum_{n=1}^{\infty} \sqrt{a_n b_n}$ is convergent.

(C) $\sum a_n b_n \leq (\sum a_n)(\sum b_n)$
$\Rightarrow \sum a_n b_n$ is convergent.

(D) $\because \sum_{n=1}^{\infty} |(-1)^n a_n b_n| = \sum_{n=1}^{\infty} a_n b_n$ is convergent,
$\Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n b_n$ is convergent.

13. (A) Let $a_n = \frac{5}{4n-3\ln n-7}$, $b_n = \frac{1}{n}$.
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{5n}{4n-3\ln n-7} = \frac{5}{4}$$
By limit comparison test, $\sum a_n$ is divergent.

(B) Let $a_n = \tan(\frac{1}{n})$, $b_n = \frac{1}{n}$,
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\tan\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\cos\frac{1}{n}}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = 1$$
By limit comparison test, $\sum a_n$ is divergent.

(C) Let $a_n = (\sqrt[n]{2}-1)^n$,
$$\lim_{n\to\infty}(|a_n|)^{1/n} = \lim_{n\to\infty}\sqrt[n]{2}-1 = 0.$$
By root test, $\sum a_n$ is convergent.

(D) Let $a_n = \frac{e^n+n}{e^{2n}-n^2}$, $b_n = \frac{1}{e^n}$,
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{e^{2n}+e^n n}{e^{2n}-n^2} = 1.$$
By limit comparision test, $\sum a_n$ is convergent.

14. (A)
$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = 0$$
$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = 0.$$

(B) Let $y = mx^2$,
$$\lim_{x\to 0, y=mx^2} f(x,y) = \lim_{x\to 0, y=mx^2}\frac{x^2(mx^2)}{x^4+m^2x^4} = \frac{m}{1+m^2}$$
$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.
$\Rightarrow f$ is discontinuous at $(0,0)$.

(C) By (B), f is not differentiable at $(0,0)$.

(D)
$$D_{<\frac{\sqrt{3}}{2},\frac{1}{2}>}f(0,0) = \lim_{h\to 0}\frac{f(0+\frac{\sqrt{3}}{2}h, 0+\frac{1}{2}h)-f(0,0)}{h} = \frac{3}{2}$$

15. By definition, choose (B) and (C).

1.
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{d\theta} = r'\cos\theta + r(-\sin\theta) \\ \dfrac{dy}{d\theta} = r'\sin\theta + r\cos\theta \end{cases}$$

Then

$$\begin{aligned}
\text{Length} &= \int_0^{\frac{\pi}{4}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta \\
&= \int_0^{\frac{\pi}{4}} \sqrt{(r')^2 + r^2} d\theta \\
&= \int_0^{\frac{\pi}{4}} \sqrt{(3\cos^2(\frac{\theta}{3})(-\sin\frac{\theta}{3})\frac{1}{3})^2 + (\cos^3(\frac{\theta}{3}))^2} d\theta \\
&= \int_0^{\frac{\pi}{4}} \sqrt{(\cos^4\frac{\theta}{3})(\sin\frac{\theta}{3} + \cos^2\frac{\theta}{3})} d\theta \\
&= \int_0^{\frac{\pi}{4}} \cos^2\frac{\theta}{3} d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1 + \cos\frac{2\theta}{3}}{2} d\theta \\
&= \frac{1}{2}\left(\theta + \frac{3}{2}\sin\frac{2}{3}\theta \Big|_0^{\frac{\pi}{4}}\right) \\
&= \frac{3+\pi}{8}
\end{aligned}$$

2. $f(x,y) = \dfrac{-2y}{x^2+y^2+4}$

$$\begin{cases} f_x(x,y) = \dfrac{4xy}{(x^2+y^2+4)^2} = 0 \\ f_y(x,y) = \dfrac{-2x^2+2y^2+4}{(x^2+y^2+4)^2} = 0 \end{cases} \Rightarrow \begin{cases} 4xy = 0 \\ -2x^2+2y^2+4 = 0 \end{cases}$$

$\Rightarrow (x,y) = (\sqrt{2}, 0)$ or $(-\sqrt{2}, 0)$.
Consider the boundary, $x^2 + y^2 = 1$.
Let $x = \cos\theta,\ y = \sin\theta,\ f(x,y) = f(\cos\theta, \sin\theta) = \dfrac{-2\sin\theta}{5}$.
When $\theta = \pi,\ f(x,y)$ has absolute maximum $\frac{2}{5}$.

3. $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0,0)}{h} = 0$

$$f_x(x,y) = \begin{cases} 2x\tan^{-1}\frac{y}{x} + x^2\frac{1}{\sqrt{1+(\frac{y}{x})^2}}(-\frac{y}{x^2}) - y^2\frac{1}{\sqrt{1+(\frac{x}{y})^2}}(\frac{1}{y}) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0, 0+h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h\frac{1}{\sqrt{1+0}}}{h} = -1$$

4. Let $f(x,y,z) = \sin(xyz) - x - 2y - 3z$.
$\nabla f =< \cos(xyz)yz - 1, \cos(xyz)xz - 2, \cos(xyz)xy - 3 >$
$\nabla f(2,-1,0) = (-1,-2,-5)$
tangent plane: $-(x-2) - 2(y+1) - 5(z-0) = 0 \Rightarrow x + 2y + 5z = 0$

5.

$$
\begin{aligned}
& \int\int\int_E dV \\
= \quad & \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 \rho^2 \sin\phi d\rho d\phi d\theta \\
= \quad & \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\rho^3 \sin\phi d\phi d\theta \\
= \quad & \frac{8}{3}\int_0^{2\pi} -\cos\phi\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \\
= \quad & \frac{8\sqrt{2}}{3}\pi
\end{aligned}
$$