一百零五學年度第一學期微積分會考試題解答 (A 卷)

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December 10, 2017

1. $f(x) = x - 1 - \dfrac{1-x}{x(x+1)}$, so

$$\lim_{x\to 0^+} f(x) = 0 - 1 - \frac{1}{1}\cdot\left(\lim_{x\to 0^+}\frac{1}{x}\right) = -\infty,$$

and

$$\lim_{x\to -1^-} f(x) = -\infty = -1 - 1 - \left(\frac{2}{-1}\right)\cdot\left(\lim_{x\to -1^-}\frac{1}{x+1}\right) = -\infty.$$

The vertical asymptotes are $x = 0$, $x = -1$. Since $\lim_{x\to\pm\infty}\dfrac{1-x}{x(x+1)} = 0$, the slant asymptote is $y = x - 1$ and there is no horizontal asymptote.

(Actually, $\lim_{x\to 0^-} f(x) = \lim_{x\to -1^+} f(x) = \infty$.)

`Ans: (B).`

2. $f'(x) = \dfrac{[g'(h(x))h'(x)]\,h(x) - g(h(x))h'(x)}{(h(x))^2}$, so

$$f'(6) = \frac{g'(h(6))h'(6)h(6) - g(h(6))h'(6)}{(h(6))^2} = \frac{g'(3)\cdot(-2)\cdot 3 - g(3)\cdot(-2)}{3^2} = \frac{-42+6}{9} = -4.$$

`Ans: (D).`

3. Let $g(x) = x^3 - 4x + 1$, then $f(x) = \ln|g(x)|$, by chain rule, we have

$$f'(x) = \frac{g'(x)}{g(x)} = \frac{3x^2-4}{x^3-4x+1}.$$

`Ans: (A).`

4. Notice that

$$\left(1 - \frac{1}{x} - \frac{2}{x^2}\right)^x = \exp\left(x\ln\left(1 - \frac{1}{x} - \frac{2}{x^2}\right)\right).$$

Let $u = 1/x$, then

$$\begin{aligned}\lim_{x\to\infty} x\ln\left(1 - \frac{1}{x} - \frac{2}{x^2}\right) &= \lim_{u\to 0^+}\frac{\ln(1-u-2u^2)}{u}\\ &= \lim_{u\to 0^+}\frac{-1-4u}{1-u-u^2} \qquad \text{(L'Hopital rule of type 0/0)}\\ &= -1.\end{aligned}$$

Hence $\lim_{x\to\infty}\left(1 - \dfrac{1}{x} - \dfrac{2}{x^2}\right)^x = \exp(-1)$.

`Ans: (B).`

5. $f''(x) = 6 \cdot 5x^4 - 15 \cdot 2 = 30x^4 - 30 = 30\,(x^2+1)\,(x^2-1) = 30\,(x^2+1)\,(x+1)(x-1) = 0$, then $x = \pm 1$. Because $f''(x) > 0$ for $x < -1$ or $x > 1$, on the other hand, $f''(x) < 0$ for $-1 < x < 1$. Hence the inflection points of f are $(1, f(1))$ and $(-1, f(-1))$.

`Ans: (C).`

6. Let $u = e^x$, then $du = e^x dx$, hence

$$\int_0^{(\ln 3)/2} e^x \sqrt{1+e^{2x}}\, dx = \int_1^{\sqrt{3}} \sqrt{1+u^2}\, du$$
$$= \left(\frac{u\sqrt{1+u^2}}{2} + \frac{\ln\left(u+\sqrt{1+u^2}\right)}{2} \right) \Bigg|_{u=1}^{\sqrt{3}}$$
$$= \frac{1}{2}\left[2\sqrt{3} - \sqrt{2} + \ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}} \right) \right].$$

`Ans: (A).`

7. The volume of resulting solid is

$$\int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \left(-\frac{\pi}{x}\right)\Big|_{u=1}^{\infty} = \pi < \infty,$$

and its surface area is

$$\int_1^\infty 2\pi \cdot \frac{1}{x} \sqrt{1 + \left[\left(\frac{1}{x}\right)'\right]^2}\, dx = \int_1^\infty 2\pi \cdot \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx.$$

Since

$$\int_1^\infty 2\pi \cdot \frac{1}{x} \sqrt{1 + \left[\left(\frac{1}{x}\right)'\right]^2}\, dx > \int_1^\infty 2\pi \cdot \frac{1}{x}\, dx = \infty,$$

we conclude that the volume is finite, and the surface area is infinite.

`Ans: (C).`

8.

$$\int_0^{2\pi} \pi(1-\cos x)^2\, dx = \int_0^{2\pi} \pi(1+\cos^2 x - 2\cos x)\, dx$$
$$= \int_0^{2\pi} \pi(1 + \frac{\cos(2x)+1}{2} - 2\cos x)\, dx = \pi\left(\frac{3x}{2} + \frac{\sin(2x)}{4} - 2\sin x \right)\Bigg|_0^{2\pi} = 3\pi^2.$$

`Ans: (C).`

9. $x'(t) = -3\sin t + 3\sin(3t)$, $y'(t) = 3\cos t - 3\cos(3t)$, so

$$(x'(t))^2+(y'(t))^2 = 18-18\sin t\sin(3t)-18\cos t\cos(3t) = 18-18\cos(2t) = 18-18(1-2\sin^2 t) = 36\sin^2 t.$$

The length is

$$\int_0^\pi \sqrt{(x'(t))^2+(y'(t))^2}\, dt = \int_0^\pi 6\,|\sin t|\, dt = -6\cos x\,|_0^\pi = 6-(-6) = 12.$$

`Ans: (C).`

10.
$$V(t)=\int_0^1 2\pi x\ln(x+1)\,dx+\int_1^t 2\pi x(\ln(x+1)-\ln x)\,dx,$$
then by Fundamental theorem of calculus,
$$V'(t)=2\pi t\,(\ln(t+1)-\ln t)=2\pi t\ln\left(1+\frac{1}{t}\right).$$
Substitute $u=1/t$, we find the limit by l'Hôpital's Rule,
$$\lim_{t\to\infty}V'(t)=\lim_{u\to 0^+}2\pi\frac{\ln(1+u)}{u}=\lim_{u\to 0^+}2\pi\frac{1}{1+u}=2\pi.$$
`Ans: (B).`

11. $f'(x)=\dfrac{a(x^2+b^2)-2ax^2}{(x^2+b^2)^2}$. Because f has a local minimum at $x=-2$,
$$f'(-2)=\frac{a((-2)^2+b^2)-2a(-2)^2}{((-2)^2+b^2)^2}=\frac{4a+ab^2-8a}{(4+b^2)^2}=0,\quad ab^2-4a=0,$$
and $f'(0)=\dfrac{ab^2}{b^4}=\dfrac{a}{b^2}=1$. It implies $a=4$, $b=\pm 2$.
`Ans: (A)(B).`

12. First, for $x\neq 0,9$,
$$\begin{aligned}f'(x)&=\frac{2}{3}x^{-1/3}(9-x)^{1/3}-\frac{1}{3}x^{2/3}(9-x)^{-2/3}\\&=\frac{1}{3}x^{-1/3}(9-x)^{-2/3}(2(9-x)-x)\\&=\frac{1}{3}x^{-1/3}(9-x)^{-2/3}(18-3x).\end{aligned}$$
It is obvious that $f'(x)>0$ on $(0,6)$ and $f'(x)<0$ on $(-\infty,0)\cup(6,9)\cup(9,\infty)$, critical number are $x=0,6,9$. By information, f has local maximum at $x=6$ and local minimum at $x=0$.
`Ans: (A)(C)(D).`

13. (A) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1\neq f(0)$, so f is not continuous at 0.
(B) $\sin x$ and x are differentiable function and not equal to 0 on $(0,1)$, so f is a differentiable function on $(0,1)$.
(C)(D) Since f and $f\circ f$ are continuous on $(0,1]$, and f and $f\circ f$ are bounded on $[0,1]$, thus both f and $f\circ f$ are integrable on $[a,b]$.
`Ans: (B)(C)(D).`

14. (A) $x^3-x=\sin(\pi x)$ implies $x=0,1$.
(B) For $x\in(0,1)$, $\sin(\pi x)>0>x^3-x$, so the area of R is $\displaystyle\int_0^1 \sin(\pi x)-(x^3-x)\,dx$.
(C) Let $A(t)$ is the area of equilateral triangle at $x=t$, then $A(t)=(\sin(\pi t)-t^3+t)^2\cdot\frac{3}{4}$. Hence the volume is $\int_0^1 A(t)dt$.
(D) Use shell method to compute the volume. The radius of cylinder is $(x-(-1))=x+1$, and the height of cylinder is $\sin(\pi x)-x^3+x$, so the volume is $\displaystyle\int_0^1 2\pi(x+1)(\sin(\pi x)-x^3+x)\,dx$.
`Ans: (B)(C)(D).`

15. By Fundamental Theorem of Calculus, $f'(x) = (x^2 - 4x + 3)e^{-x}$.
(A)(B) $f'(x) = (x-3)(x-1)e^{-x}$, $f'(x) > 0$ if and only if $(x-3)(x-1) > 0$, if and only if $x > 3$ or $x < 1$. So f is increasing on $x > 3$ or $x < 1$.
(C)(D) $f'(x) \le 0$ on $[1,3]$ and $f(1) = 0$, so $f(3) < 0$.
`Ans: (B)(D).`

1. Use 2017 times l'Hôpital's Rule (all limits are of type (∞/∞)), we have

$$\lim_{x\to\infty} \frac{x^{2017}}{2^x} = \lim_{x\to\infty} \frac{2017!}{2^x(\ln 2)^{2017}} = \frac{2017!}{\infty} = 0.$$

`Ans:` 0.

2. Take implict differentiation, we have $2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$. Substitute $(x,y) = (0, 1/2)$, then we have

$$\begin{aligned}
&2\cdot 0 + 2\cdot\frac{1}{2}\cdot\left(y'\,|_{(x,y)=(0,1)}\right)\\
&= 2\left(2\cdot 0^2 + 2\cdot\left(\frac{1}{2}\right)^2 - 0\right)\cdot\left(4\cdot 0 + 4\cdot\left(\frac{1}{2}\right)\cdot\left(y'\,|_{(x,y)=(0,1)}\right) - 1\right)\\
&= 2\cdot\left(y'\,|_{(x,y)=(0,1)}\right) - 1
\end{aligned}$$

so $y'\,|_{(x,y)=(0,1)} = 1$, the tangent line is $y = x + 1/2$.
`Ans:` $y = x + \frac{1}{2}$.

3. First, find all critical number of f.

$$f'(x) = \sqrt{9-x^2} + \left(\frac{-2x^2}{2\sqrt{9-x^2}}\right) = \frac{1}{\sqrt{9-x^2}}((9-x^2)-x^2) = \frac{1}{\sqrt{9-x^2}}(9-2x^2) = 0,$$

so critical number are $\pm\frac{3}{\sqrt{2}}$. Second, verify the critical number and all endpoint. $f(-3) = f(3) = 0$, $f\left(\frac{3}{\sqrt{2}}\right) = \frac{9}{2} = -f\left(-\frac{3}{\sqrt{2}}\right)$, so the absolute maximum is $\frac{9}{2}$.
`Ans:` $\frac{9}{2}$.

4. By Fundamental Theorem of Calculus, $f'(x) = e^{x^2}$, so $f''(x) = 2xe^{x^2}$ by Chain Rule.
`Ans:` $2xe^{x^2}$.

5. Notice that

$$0 < \int_1^\infty x^p|\sin x|\,dx < \int_1^\infty x^p\,dx.$$

Because $\int_1^\infty x^p\,dx$ is convergent for $p < -1$, $\int_1^\infty x^p|\sin x|dx$ is also convergent for $p < -1$.
`Ans:` $p < -1$.