一百零五學年度第二學期微積分會考試題解答 (A 卷)

林峻緯

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[單選擇題]

1.

$$\mathbf{r}_1(s) = \mathbf{r}_2(t) \Leftrightarrow \begin{cases} s^2+3=t^2, \\ -s+1=t-2, \\ s=-t+3. \end{cases} \Leftrightarrow \begin{cases} (-t+3)^2+3=t^2, \\ s=-t+3. \end{cases} \Leftrightarrow \langle s,t\rangle = \langle 1,2\rangle .$$

So the intersection point is $\mathbf{r}_1(s) = \mathbf{r}_2(t) = \langle 4,0,1\rangle$. Now,

$$\mathbf{r}_1'(1) = \langle 2s,-1,1\rangle\big|_{s=1} = \langle 2,-1,1\rangle ; \quad \mathbf{r}_2'(2) = \langle 2t,1,-1\rangle\big|_{t=2} = \langle 4,1,-1\rangle .$$

The normal vector of tangent plane is parallel to $\mathbf{r}_1'(1)\times\mathbf{r}_2'(2) = \langle 0,6,6\rangle$, which is also parallel to $\langle 0,1,1\rangle$. So $\langle 0,1,1\rangle\cdot\langle x-4,y-0,z-1\rangle = 0$, that is, $y+z-1=0$.
`Ans: (A).`

2. `(A)` By l'Hôpital's Rule, we know that

$$\lim_{x\to\infty}\frac{x}{(1.5)^x} = 0.$$

Thus for sufficient large n,

$$n < (1.5)^n, \quad 0 < \frac{n}{2^n} < \frac{(1.5)^n}{2^n} < (0.75)^n.$$

We know that $\sum_{n=1}^{\infty}(0.75)^n$ converges. By comparison test, we can say that $\sum_{n=1}^{\infty}\frac{n}{2^n}$ converges.
`(B)` Since

$$a_n := \frac{\sqrt{n}}{n+1} = \sqrt{\frac{n}{(n+1)^2}} > \sqrt{\frac{n+1}{(n+2)^2}} = \frac{\sqrt{n+1}}{n+2} = a_{n+1} > 0 \quad \forall\, n\in\mathbb{N},$$

by alternating test, $\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{n}}{n+1}$ converges.
`(C)` By $p-$series test, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.
`(D)` For $n\geq 4$,

$$\frac{1}{\sqrt{n}} - \frac{1}{n} = \frac{n-\sqrt{n}}{n\sqrt{n}} = \frac{\sqrt{n}-1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} > 0.$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ converges by p-series test, by comparison test, $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{n}\right)$ diverges.
`Ans: (D).`

3.

$$f(x) = xe^x = x\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k.$$

Thus $\dfrac{f^{(100)}(0)}{100!} = \dfrac{1}{99!}$, $f^{(100)}(0) = 100$.
`Ans: (D).`

4. We first find the critical points. For $(a,b) \in \mathbb{R}$,

$$\begin{aligned} &(f_x(a,b), f_y(a,b)) = \left(2a + 2ab, 2b + a^2\right) = (0,0) \\ \Leftrightarrow &\begin{cases} 2a(b+1) = 0, \\ b = -\dfrac{a^2}{2}. \end{cases} \quad \Leftrightarrow (a,b) = (0,0) \text{ or } (\pm\sqrt{2}, -1). \end{aligned}$$

To determine the saddle points, consider

$$\begin{aligned} D(a,b) := \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} &= f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 \\ = (2+2b)\cdot 2 - (2a)^2 &= 4\left(-a^2 + b + 1\right). \end{aligned}$$

Since $D(0,0) = 4 > 0$ and $f_{xx}(a,b) = 2 + 2\cdot 0 = 2 > 0$, $(0,0)$ is a local minimum of f. For $D(\pm\sqrt{2}, -1) = -8 < 0$, we can say that $(\pm\sqrt{2}, -1)$ are saddle points of f.
`Ans: (C).`

5. Let $f(x,y,z) = x\ln y + y\ln z + xyz$. Then $\nabla f(x,y,z) = \left\langle \ln y + yz, \dfrac{x}{y} + \ln z + xz, \dfrac{y}{z} + xy \right\rangle$. $\langle a, b, -1\rangle$ should be parallel to $\nabla f(0,1,1) = \langle 1, 0, 1\rangle$. So $\langle a, b\rangle = \langle -1, 0\rangle$.
`Ans: (C).`

6.

$$\begin{aligned} \int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy &= \int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx \quad \text{(by Fubini's theorem)} \\ &= \int_0^1 \left(x^2 \cdot \frac{e^{xy}}{x}\Big|_{y=0}^{x} \right) dx = \int_0^1 \left(x\exp x^2 - x \right) dx \\ &= \frac{1}{2}\left(\exp x^2 - x^2\right)\Big|_{x=0}^{1} = \frac{e-2}{2}. \end{aligned}$$

`Ans: (A).`

7. The surface area is equal to

$$\begin{aligned} &\iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_0^1 \int_0^y \sqrt{1 + 1^2 + (2y)^2}\, dx\, dy \\ = &\int_0^1 y\sqrt{2 + 4y^2}\, dy = \frac{1}{12}\left(4y^2 + 2\right)^{3/2}\Big|_{y=0}^{1} = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{6}. \end{aligned}$$

`Ans: (D).`

8. Use double integral in polar coordinate,

$$\begin{aligned}
&\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2}\,dy\,dx \\
&= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta + r\sin\theta}{(r\cos\theta)^2 + (r\sin\theta)^2}\, r\,dr\,d\theta = \int_0^{\pi/2} \left[(\cos\theta + \sin\theta)\ r\,\Big|_{r=0}^{2\cos\theta} \right] d\theta \\
&= \int_0^{\pi/2} \left(2\cos^2\theta + 2\cos\theta\sin\theta\right) d\theta = \int_0^{\pi/2} \left(\cos(2\theta) + 1 + \sin(2\theta)\right) d\theta \\
&= \frac{\sin(2\theta) - \cos(2\theta)}{2} + \theta \Bigg|_{\theta=0}^{\pi/2} = \frac{\pi}{2} + 1.
\end{aligned}$$

`Ans: (C).`

9. Use triple integral in polar coordinate,

$$\begin{cases} x = r\sin\phi\cos\theta, \\ y = r\sin\phi\sin\theta, \\ z = r\cos\phi, \end{cases}$$

for $r \geq 0$, $0 \leq \phi \leq \pi$, and $0 \leq \theta < 2\pi$. Then by changing of variable of spherical coordinate, the triple integral is equal to

$$\begin{aligned}
&\int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{r^2\sin\phi}{(r^2+1)^2}\,d\phi\,d\theta\,dr = 4\pi \int_0^\infty \frac{r^2}{(r^2+1)^2}\,dr \\
&= 4\pi \int_0^{\pi/2} \frac{\tan^2\alpha}{(\tan^2\alpha+1)^2}\sec^2\alpha\,d\alpha \qquad (\text{let } r = \tan\alpha,\ dr = \sec^2\alpha\,d\alpha) \\
&= 4\pi \int_0^{\pi/2} \frac{\tan^2\alpha}{\sec^2\alpha}\,d\alpha = 4\pi \int_0^{\pi/2} \sin^2\alpha\,d\alpha = 4\pi \int_0^{\pi/2} \frac{1-\cos(2\alpha)}{2}\,d\alpha \\
&= 4\pi \cdot \frac{2\alpha - \sin(2\alpha)}{4} \Bigg|_{\alpha=0}^{\pi/2} = 4\pi \cdot \frac{\pi}{4} = \pi^2.
\end{aligned}$$

`Ans: (B).`

10. The bounded solid can be performed as the following type,

$$\begin{aligned}
D &= \left\{(x,y,z) \mid 0 \leq x^2 \leq y \leq 4,\ 0 \leq z \leq y\right\} \\
&= \left\{(x,y,z) \mid 0 \leq z \leq y,\ x^2 \leq y \leq 4,\ -2 \leq x \leq 2\right\} \\
&= \{(x,y,z) \mid 0 \leq z \leq y,\ -\sqrt{y} \leq x \leq \sqrt{y},\ 0 \leq y \leq 4\} \\
&= \left\{(x,y,z) \mid z \leq y \leq 4,\ x^2 \leq z \leq 4,\ -2 \leq x \leq 2\right\} \cup \left\{(x,y,z) \mid x^2 \leq y \leq 4,\ 0 \leq z \leq x^2,\ -2 \leq x \leq 2\right\}.
\end{aligned}$$

`Ans: (C).`

[多選擇題]

11. `(A)` Let $\sum_{n=0}^\infty b_n = \beta \in \mathbb{R}$. Since $b_n > 0$, $0 \leq b_n \leq \beta$ for all $n \in \mathbb{N}$. Hence $0 \leq a_n b_n \leq \beta\, a_n$. If $\sum_{n=0}^\infty a_n$ converges, then $\sum_{n=0}^\infty (\beta\, a_n) = \beta \cdot \sum_{n=0}^\infty a_n$ does also. By comparison test, $\sum_{n=0}^\infty a_n b_n$ converges.

`(B)` Since $|(-1)^n a_n| = a_n > 0$ and $\sum_{n=0}^\infty a_n$ converges, we have that $\sum_{n=0}^\infty (-1)^n a_n$ converges absolutely.

`(C)` Set $a_n = \dfrac{1}{(n+1)^2}$, $b_n = \dfrac{1}{n+1}$ for all $n \in \mathbb{N}$. Then $a_n, b_n > 0$ for all $n \in \mathbb{N}$. $\sum_{n=0}^\infty a_n = \sum_{k=1}^\infty \dfrac{1}{k^2}$

converges by p-series test, and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{1}{n+1} = 0$. But $\sum_{n=0}^{\infty} b_n = \sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges by p-series test.

(D) Since $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$ by divergence test. Notice that $1 + a_n > 1 + 0 = 1$, $\ln(1 + a_n) > \ln 1 = 0$. Apply l'Hôpital's Rule to find the limit,

$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{t\to 0^+} \frac{\ln(1+t)}{t} = \lim_{t\to 0^+} \frac{1}{1+t} = 1 \in (0, \infty).$$

With the hypothesis that $\sum_{n=0}^{\infty} a_n$ converges, we have $\sum_{n=0}^{\infty} \ln(1+a_n)$ converges by limit comparison test.

Ans: (A)(B)(D).

12. (A) Since $\left|\dfrac{f^{(n)}(0)}{n!}\right| \le \dfrac{1}{n!}$ for all $x \in (-1, 1)$, and $\sum_{n=0}^{\infty} \dfrac{1}{n!} = e$, by comparison test, $\sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$ converges absolutely for $x \in (-1, 1)$.

(B) Any power series $\sum_{n=0}^{\infty} a_n x^n$ whose interval of convergence must of one of the following forms:

$$x_0,\ (x_0 - r, x_0 + r),\ (x_0 - r, x_0 + r],\ [x_0 - r, x_0 + r),\ [x_0 - r, x_0 + r],\ (-\infty, \infty), \quad x_0 \in \mathbb{R}, r > 0.$$

(C) $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n} x^n$ converges for $x = 1$ by alternating test. But $\sum_{n=0} \dfrac{(-1)^n}{n} (-1)^n = \sum_{n=0}^{\infty} \dfrac{1}{n} = \infty$.

(D) Let $y = x^2$. Then we have $\sum_{n=0}^{\infty} a_n x^{2n} = \sum_{n=0}^{\infty} a_n y^n$. Hence the radius of convergence of $\sum_{n=0}^{\infty} a_n x^{2n}$ is $\sqrt{2}$.

Ans: (A)(D).

13. For $x \neq -\dfrac{1}{2}$, let $y = \dfrac{x}{2x+1}$. Then $\sum_{n=0}^{\infty} \dfrac{nx^n}{(n+1)(2x+1)^n} = \sum_{n=0}^{\infty} \dfrac{n}{n+1} y^n$. Here, we set $a_n = \dfrac{n}{n+1} y^n$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} |y| = |y|.$$

Therefore, $\sum_{n=0}^{\infty} \dfrac{nx^n}{(n+1)(2x+1)^n} = \sum_{n=0}^{\infty} \dfrac{n}{n+1} y^n$ converges absolutely while $|y| < 1$; that is, $\left|\dfrac{x}{2x+1}\right| < 1$. Thus $-1 < \dfrac{x}{2x+1} < 1$, $x > -\dfrac{1}{3}$ or $x < -1$. For $|y| = 1$, $\sum_{n=0}^{\infty} \dfrac{n}{n+1} y^n$ diverges since $\lim_{n\to\infty} \dfrac{n}{n+1} = 1 \neq 0$. Finally, we can certainly say that $\sum_{n=0}^{\infty} \dfrac{nx^n}{(n+1)(2x+1)^n}$ converges absolutely if and only if $x \in (-\infty, -1) \cup \left(-\dfrac{1}{3}, \infty\right)$.

Ans: (B)(C)(D).

14. (A) Let $\theta \in \mathbb{R}$. Set $\vec{u} = (\cos\theta, \sin\theta)$.

Case I: $\sin\theta = 0$. Then

$$D_{\vec{u}}(0,0) = \lim_{r\to 0} \frac{f(r\cos\theta, r\sin\theta) - f(0,0)}{r} = \lim_{r\to 0} \frac{(r^2\cos^2\theta)(r\sin\theta)}{r(r^4\cos^4\theta + r^2\sin^2\theta)} = 0.$$

Case II: $\sin\theta \neq 0$. Then

$$D_{\vec{u}}(0,0) = \lim_{r\to 0} \frac{\cos^2\theta\sin\theta}{(r^2\cos^4\theta + \sin^2\theta)} = \frac{\cos^2\theta}{\sin\theta}.$$

Thus $D_{\vec{u}}(0,0)$ exist for all unit vector $\vec{u}$. But

$$\lim_{t\to 0} f(t,t^2) = \lim_{t\to 0} \frac{t^2 \cdot t^2}{t^4 + (t^2)^2} = \frac{1}{2} \neq f(0,0),$$

f is discontinuous at $(0,0)$.

(B) For $\vec{u} \neq \pm(1,0)$ or $\pm(0,1)$, we have

$$D_{\vec{u}}(0,0) = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r(r^2\cos^2\theta + r^2\sin^2\theta)} = \lim_{r\to 0} \frac{\cos\theta\sin\theta}{r} \text{ diverges.}$$

(C)

$$\begin{aligned} D_{\vec{u}}(0,0) &= \lim_{r\to 0} \frac{(2r^2\cos^2\theta + r^4\cos^4\theta)(r\sin\theta)}{r[(2r^2\cos^2\theta + r^4\cos^4\theta)^2 + r^2\sin^2\theta]} \\ &= \lim_{r\to 0} \frac{(2\cos^2\theta + r^2\cos^4\theta)\sin\theta}{(2r\cos^2\theta + r^3\cos^4\theta)^2 + \sin^2\theta} \\ &= \begin{cases} 0, \text{ if } \sin\theta = 0, \\ \dfrac{2\cos^2\theta}{\sin\theta}, \text{ if } \sin\theta \neq 0. \end{cases} \end{aligned}$$

But

$$\lim_{t\to 0} f(t, 2t^2 + t^4) = \lim_{t\to 0} \frac{(2t^2+t^4)\cdot(2t^2+t^4)}{(2t^2+t^4)^2 + (2t^2+t^4)^2} = \frac{1}{2} \neq f(0,0),$$

f is discontinuous at $(0,0)$.

(D) Any polynomial are differentiable on $\mathbb{R}^2$, and thus, directional derivatives exist and keep the property of continuity.

Ans: (A)(C)

15. (A) By Fubini's theorem,

$$\begin{aligned} F(x,y) &= \int_0^x \int_0^{2y} \sin(st)\,dt\,ds = \int_0^x \left(-\frac{1}{s}\cos(st)\Big|_{t=0}^{2y} \right) ds \\ &= \int_0^x \frac{1}{s}[1-\cos(2ys)]\,ds \quad \text{for all } (x,y) \in \mathbb{R} \end{aligned} \tag{1}$$

By fundamental theorem of calculus, we have

$$F_x(x,y) = \frac{1}{x}[1-\cos(2xy)] \quad \text{for } x \neq 0 \text{ and for all } y \in \mathbb{R}.$$

(B) By Fubini's theorem,

$$\begin{aligned} F(x,y) &= \int_0^{2y} \int_0^{x} \sin(st)\,ds\,dt = \int_0^{2y} \left(-\frac{1}{t}\cos(st)\Big|_{s=0}^{x} \right) dt \\ &= \int_0^{2y} \frac{1}{t}[1-\cos(xt)]\,dt \quad \text{for all } (x,y) \in \mathbb{R} \end{aligned} \tag{2}$$

By fundamental theorem of calculus, we have

$$F_y(x,y) = \left(\frac{d}{dy}(2y)\right)\left(\frac{1}{2y}[1-\cos(2xy)]\right) = \frac{1}{y}[1-\cos(2xy)] \quad \text{for } y \neq 0 \text{ and for all } x \in \mathbb{R}.$$

(C) Now, by definition of partial derivative, we obtain

$$F_x(0,0) = \lim_{t\to 0} \frac{F(t,0) - F(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0 \quad \text{by (1).}$$

(D) By (A),

$$F_{xy}(x,y) = \left(\frac{\partial}{\partial y}(2xy)\right)\left(\frac{1}{x}\sin(2xy)\right) = 2\sin(2xy) \quad \text{for } x \neq 0 \text{ and for } y \neq 0.$$

On the other hand,

$$F_{yx}(x,y) = \left(\frac{\partial}{\partial x}(2xy)\right)\left(\frac{1}{y}\sin(2xy)\right) = 2\sin(2xy). \quad \text{for } x \neq 0 \text{ and } \text{ for } y \neq 0.$$

`Ans: (B)(D)`

[填空題]

1. The point whose polar coordinate $(r,\theta) = \left(\frac{1}{2}, \frac{2\pi}{3}\right)$ is $(r\cos\theta, r\sin\theta) = \left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$. The slope of the tangent line at $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ is

$$\frac{dy}{dx}\bigg|_{(x,y)=(-1/4,\sqrt{3}/4)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}\Bigg|_{(r,\theta)=(1/2,2\pi/3)}$$
$$= \frac{-\sin^2\theta + r\cos\theta}{-\sin\theta\cos\theta - r\sin\theta}\bigg|_{(r,\theta)=(1/2,2\pi/3)} \quad \text{does not exist.}$$

Thus, the tangent line is vertical, we can say that $b = 0$. Put $(x,y) = \left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$ into $ax+by+1 = 0$, we obtain that $a = 4$.
`Ans:` $(4,0)$.

2.
$$3\sin\theta = 1 + \sin\theta \;\Leftrightarrow\; \sin\theta = \frac{1}{2} \;\Leftrightarrow\; \theta = \frac{\pi}{6} + 2k\pi \text{ or } \theta = \frac{5\pi}{6} + 2k\pi,\ k \in \mathbb{Z}.$$

So those curves intersect at point of polar coordinate $(r,\theta) = \left(\frac{3}{2}, \frac{\pi}{6}\right)$ or $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.

Notice that $3\sin\theta \geq 1 + \sin\theta$ if and only if $\frac{\pi}{6} + 2k\pi \leq \theta \leq \frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$. Hence, the area of the region that lies inside both curves is

$$\int_0^{\pi/6} \frac{1}{2}(3\sin\theta)^2\,d\theta + \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2\,d\theta + \int_{5\pi/6}^{\pi} \frac{1}{2}(3\sin\theta)^2\,d\theta$$
$$= 2\left(\int_0^{\pi/6} \frac{1}{2}(3\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(1+\sin\theta)^2\,d\theta\right)$$

(by the symmetry of those curves with respect to y- axis)

$$= 2\left(\left(\frac{9\theta}{4} - \frac{9\sin(2\theta)}{8}\right)\bigg|_{\theta=0}^{\pi/6} + \left(\frac{3\theta}{4} - \cos\theta - \frac{\sin(2\theta)}{8}\right)\bigg|_{\theta=\pi/6}^{\pi/2}\right) = \frac{5\pi}{4}.$$

`Ans:` $\frac{5\pi}{4}$.

3. Let $g(x,y,z) = x^2 + y^2 + \frac{z^2}{2} - 6$, and $\lambda \in \mathbb{R}$. If (a,b,c) is on the constraint such that $f(a,b,c)$ reaches the maximum, then

$$\nabla f(a,b,c) + \lambda \nabla g(a,b,c) = (b, a-c, -b) + \lambda(2a, 2b, c)$$
$$= (2\lambda a + b, a + 2\lambda b - c, -b + \lambda c) = (0,0,0),$$

and $g(a,b,c) = 0$. Next, we will find the value a, b, c.

`Case I:` $\lambda = 0$.

Then $(b, a-c, -b) = (0,0,0)$, $f(a,b,c) = b(a-c) = 0$.

`Case II:` $\lambda \neq 0$.

Then $(2\lambda a + b) + (-b + \lambda c) = \lambda(2a + c) = 0$, $c = -2a$. And so on, $b = \lambda c = -2\lambda a$. Therefore, $a + 2\lambda b - c = a + 2\lambda(-2\lambda a) - (-2a) = (-4\lambda^2 + 3)a = 0$.

`Subcase I:` $a = 0$.

Thus $b = c = 0$. But $g(a,b,c) = -6 \neq 0$, contradiction.

`Subcase II:` $a \neq 0$.

Hence $-4\lambda^2 + 3 = 0$, $\lambda = \pm\frac{\sqrt{3}}{2}$. Now, $g(a,b,c) = a^2 + (-2\lambda a)^2 + \frac{(-2a)^2}{2} - 6 = 6a^2 - 6 = 0$, $a = \pm 1$. Therefore, $f(a,b,c) = b(a-c) = (-2\lambda a)[a - (-2a)] = -6\lambda a = \pm 3\sqrt{3}$.
`Ans:` $3\sqrt{3}$.

4. Let $(a,b) = (1,0)$ and $(h,k) = (0.2, 0.1)$. For convenience, we first get $u(a,b) = 1$ and $v(a,b) = \frac{\pi}{4}$. Then

$$\begin{aligned}
f(a+h, b+k) &\sim f(a,b) + \nabla f(a,b) \cdot (h,k) \\
&= g(u(a,b), v(a,b)) + \left((g_u u_x + g_v v_x,\ g_u u_y + g_v v_y)\,|_{(x,y)=(a,b)} \right) \cdot (h,k) \\
&= g\left(1, \frac{\pi}{4}\right) + \left(\left(\frac{v}{u} \cdot 1 + (\ln u) \cdot \frac{e^y}{1+x^2},\ \frac{v}{u} \cdot 2y + (\ln u) \cdot e^y \arctan x \right) \Big|_{(x,y)=(a,b)} \right) \cdot (h,k) \\
&= \frac{\pi}{4} \cdot (\ln 1) + \left(\frac{\pi}{4}, 0 \right) \cdot (0.2, 0.1) = \frac{0.2\pi}{4} = 0.05\pi.
\end{aligned}$$

`Ans:` 0.05.

5. Notice that $R = \{(x,y) \mid \pi \leq x + y \leq 3\pi, -\pi \leq x - y \leq \pi\}$. Let $u = x + y$ and $v = x - y$. Then $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. By Fubini's theorem and changing of variables, the integral is equal to

$$\int_{-\pi}^{\pi} \int_{\pi}^{3\pi} (v^2 \sin^2 u) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| du\, dv = \frac{1}{4} \left(\int_{-\pi}^{\pi} v^2\, dv \right) \left(\int_{\pi}^{3\pi} \sin^2 u\, du \right) = \frac{1}{4} \cdot \frac{2\pi^3}{3} \cdot 2\pi = \frac{\pi^4}{3}.$$

`Ans:` $\frac{\pi^4}{3}$.