# 一百零六學年度第一學期微積分會考試題參考詳解

1. By implicit differentiation,

$$\frac{d}{dx}\left(x^3+y^3-9xy\right)=3x^2+3y^2\frac{dy}{dx}-9y-9x\frac{dy}{dx}=0,$$
$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,4)}=\left.\frac{x^2-3y}{3x-y^2}\right|_{(x,y)=(2,4)}=\frac{4}{5}.$$

`Ans: B`

2. $y=1-\cos\theta=0 \;\Leftrightarrow\; \theta=0,2\pi \;\Leftrightarrow\; x=\theta-\sin\theta=0,2\pi$. So the volume $V=$

$$\int \pi y^2\ dx=\int_0^{2\pi}\pi(y(t))^2\,x'(t)\ dt=\pi\int_0^{2\pi}(y(t))^3\ dt$$
$$=\pi\int_0^{2\pi}(1-\cos t)^3\ dt=\pi\int_0^{2\pi}1-3\cos t+3\cos^2 t-\cos^3 t\ dt.$$

By Lemma C.R.7,

$$V=\pi\int_0^{2\pi}1+3\cos^2 t\,dt=\pi\int_0^{2\pi}1+3\cdot\frac{\cos(2t)+1}{2}\ dt$$
$$=\pi\int_0^{2\pi}\left(1+\frac{3}{2}\right)\ dt=\pi\cdot 2\pi\cdot\frac{5}{2}=5\pi^2.$$

[In-depth Coverage]

Here is Lemma C.R.7 (Cosine Ruduction):

For nonzero integer $k$ and *odd* positive integer $n$,

$$I:=\int_0^{2\pi}(\cos(k\theta))^n\ d\theta=0.$$

[Proof]

Notice that $\cos(k\theta)=\cos\left(k\left(\theta+\frac{2\pi}{k}\right)\right)$ for all $\theta\in\mathbb{R}$, so

$$I=\sum_{j=1}^{k}\int_{(j-1)\frac{2\pi}{k}}^{j\frac{2\pi}{k}}(\cos(k\theta))^n\ d\theta=\sum_{j=1}^{k}\int_0^{\frac{2\pi}{k}}(\cos(k\theta))^n\ d\theta$$
$$=\int_0^{\frac{2\pi}{k}}(\cos(k\theta))^n\ k\ d\theta=\int_0^{2\pi}(\cos t)^n\ dt\quad (t:=k\theta,\ dt=k\,d\theta.)$$
$$=\int_0^{2\pi}\left(\sin\left(\frac{\pi}{2}-t\right)\right)^n\ dt=\int_{-\pi/2}^{3\pi/2}(\sin(-u))^n\ du\quad \left(-u:=\frac{\pi}{2}-t,\ du=dt.\right)$$
$$=\int_{-\pi}^{\pi}(-1)^n\sin^n t\ dt.\quad (\sin\theta=\sin(\theta+2\pi)\ \forall\ \theta\in\mathbb{R}.)$$

Since $\sin t$ is an odd function, $(\sin t)^n$ is also (remember that $n$ is an odd number). Hence

$$I = \int_{-\pi}^{\pi} (-1)^n \sin^n t \; dt = 0.$$

`Ans: C`

3. By Fundamental Theorem of Calculus, $\dfrac{dy}{dx} = \sqrt{\cos(2x)}$. The length of the curve is

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{1+(y')^2}\,dx = \int_0^{\pi/4} \sqrt{1+\cos(2x)}\,dx \\ &= \int_0^{\pi/4} \sqrt{1+(2\cos^2 x - 1)}\,dx = \int_0^{\pi/4} \sqrt{2}\cos x\,dx \\ &= \sqrt{2}\sin x\Big|_{x=0}^{\pi/4} = \sqrt{2}\cdot\frac{\sqrt{2}}{2} = 1. \end{aligned}$$

`Ans: B`

4.

$$\begin{aligned} &\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{4\cos(4t)}{-2\sin(2t)} = 0 \\ \Leftrightarrow\ &\begin{cases} \cos(4t) = 0, \\ \sin(2t) \neq 0 \end{cases} \quad \Leftrightarrow \begin{cases} 4t = \dfrac{\pi}{2} + m\pi,\ m \in \mathbb{Z}, \\ 2t \neq n\pi,\ n \in \mathbb{Z} \end{cases} \\ \Leftrightarrow\ & t = \frac{\pi}{8} + \frac{m\pi}{4},\ m \in \mathbb{Z} \ \Leftrightarrow\ (x(t), y(t)) = \left(\pm\frac{\sqrt{2}}{2}, \pm 1\right). \end{aligned}$$

`Ans: B`

5. Clearly, $f$ is differentiable on $\mathbb{R} - \{1\}$. Now, $f$ is differentiable at $x = 1$ if and only if

$$\begin{aligned} &\begin{cases} \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) \ (f \text{ must be continuous at } x = 1) \\ f'_-(1) = f'_+(1) \end{cases} \\ \Leftrightarrow &\begin{cases} ax^3\big|_{x=1} = x^2 + b\big|_{x=1} \\ \lim_{x\to1^-} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \dfrac{ax^3 - a}{x-1} = 3a \\ = \lim_{x\to1^+} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \dfrac{x^2+b-(1+b)}{x-1} = 2 \end{cases} \\ \Leftrightarrow &\begin{cases} a = 1 + b, \\ 3a = 2 \end{cases} \quad \Leftrightarrow (a, b) = \left(\frac{2}{3}, -\frac{1}{3}\right). \end{aligned}$$

`Ans: A`

6. Let $F(x) = \int_0^x \sin(t^2)\cos t\,dt$. By Fundamental Theorem of Calculus, $F'(x) = \sin(x^2)\cos x$. Apply L'Hô pital's Rule,

$$\lim_{x\to 0^+} x^{-3/2}\int_0^{\sqrt{x}} \sin\left(t^2\right)\cos t\ dt = \lim_{x\to 0^+}\frac{F(\sqrt{x})}{x^{3/2}}$$
$$= \lim_{x\to 0^+}\frac{F'(\sqrt{x})\cdot\frac{1}{2}x^{-1/2}}{\frac{3}{2}x^{1/2}} = \lim_{x\to 0^+}\frac{1}{3}\frac{\sin x\cos(\sqrt{x})}{x}$$
$$= \frac{1}{3}\cdot 1\cdot\cos(\sqrt{0}) = \frac{1}{3}.$$

`Ans: C`

7. First, notice that $0 \le |2^{-x}\sin(\pi x)| \le 2^{-x}$ and

$$\int_0^\infty 2^{-x}\,dx = \frac{1}{-\ln 2}2^{-x}\Big|_{x=0}^{\infty} = \frac{1}{\ln 2}.$$

By comparison test, $\int_0^\infty 2^{-x}\sin(\pi x)\ dx$ converges.
Now, for any integer $k$,

$$\int_{k-1}^{k} 2^{-x}\sin(\pi x)\ dx = \int_{k-1}^{k} 2^{-x}(-1)^{k-1}\sin[\pi x-(k-1)\pi]\ dx$$
$$= (-1)^{k-1}\int_{k-1}^{k} 2^{-x}\sin\{\pi[x-(k-1)]\}\ dx$$
$$= (-1)^{k-1}\int_0^1 2^{-[t+(k-1)]}\sin\pi t\ dt \qquad \text{(shifting/changing of variable)}$$
$$= (-1)^{k-1}2^{-(k-1)}\alpha = \left(-\frac{1}{2}\right)^{k-1}\alpha.$$

Hence

$$\int_0^\infty 2^{-x}\sin\pi x\,dx = \lim_{n\to\infty}\int_0^n 2^{-x}\sin\pi x\,dx$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{k-1}^{k} 2^{-x}\sin\pi x\ dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(-\frac{1}{2}\right)^{k-1}\alpha$$
$$= \lim_{n\to\infty}\alpha\cdot\frac{1\cdot[1-(-1/2)^n]}{1-(-1/2)} = \frac{2}{3}\alpha.$$

`Ans: C`

8. $L := \lim_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right) = \lim_{x\to 1}\dfrac{x\ln x - x + 1}{(x-1)\ln x}$. Since

$$\begin{cases}\lim_{x\to 1}(x\ln x - x + 1) = 1\cdot\ln 1 - 1 + 1 = 0,\\ \lim_{x\to 1}(x-1)\ln x = (1-1)\cdot\ln 1 = 0.\end{cases}$$

By L'Hô pital's rule,

$$L = \lim_{x\to 1} \frac{\ln x + x\cdot(1/x) - 1}{\ln x + (x-1)\cdot(1/x)} = \lim_{x\to 1} \frac{x\ln x}{x\ln x + x - 1}$$
$$= \lim_{x\to 1} \frac{\ln x + 1}{\ln x + 1 + 1} = \frac{\ln 1 + 1}{\ln 1 + 1 + 1} = \frac{1}{2}.$$

`Ans: B`

9. Set

$$\begin{cases} I &= \lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{1}{n+\sqrt{(i-1)i}}, \\ I_+ &= \lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{1}{n+\sqrt{(i-1)^2}}, \\ I_- &= \lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{1}{n+\sqrt{i^2}}. \end{cases}$$

Then $I_- \le I \le I_+$. Moreover,

$$I_+ = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n + n\sqrt{\frac{(i-1)^2}{n^2}}}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{1}{1+\sqrt{\frac{(i-1)^2}{n^2}}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{1}{1+\frac{i-1}{n}} = \int_0^1 \frac{dx}{1+x}.$$

Similarly, $I_- = \int_0^1 \dfrac{dx}{1+x}$. Hence

$$I = I_+ = I_- = \int_0^1 \frac{dx}{1+x} = \ln(1+x)\big|_{x=0}^{1} = \ln 2.$$

`Ans: B`

10. Let $f(x) = 5x + 3\sin x + 9$. Then $f'(x) = 5 + 3\cos x \in [5-3, 5+3]$ for all real $x$. So $f'(x) > 0$ for all real $x$, $f$ is strictly increasing on the real line. That is, $f$ has **at most** one root.

Now,

$$\begin{cases} f(-3) \le 5\cdot(-3) + 3\cdot 1 \quad + 9 &= -3 < 0, \\ f(-1) \ge 5\cdot(-1) + 3\cdot(-1) + 9 &= \quad 1 > 0. \end{cases}$$

By Intermediate Value Theorem, there is a real number $c$ between $-3$ and $1$ such that $f(c) = 0$.

`Ans: B`

11. From integration by parts, we have that

$$I := \int_0^1 xf'(1-x)\ dx = x\cdot[-f(1-x)]\big|_{x=0}^{1} - \int_0^1 -f(1-x)\ dx$$
$$= -f(0) + \int_0^1 f(1-x)\ dx.$$

Now, by changing of variable, set $t = 1-x$, then $dt = -dx$, and

$$I = -f(0) + \int_1^0 f(t)\cdot(-1)\ dt$$
$$= -f(0) + \int_0^1 f(t)\ dt = -f(0) = f(-0) = 0.$$

`Ans: CD`

12.

(A) Let $F(x) = \displaystyle\int_0^x \frac{dt}{1+t^4}\ \forall\ x\in\mathbb{R}$. Clearly, $F(x)$ and $x^2$ are both continuous on $\mathbb{R}$, $f(x) = F(x^2)$ is also.

(B) For any $\delta \in [-1,1]\backslash\{0\}$,

$$f(\delta) = \int_0^{\delta^2} \frac{dt}{1+t^4} =\geq \int_0^{\delta^2}\frac{dt}{1+1^4} = \frac{\delta^2}{2} > 0 = f(0).$$

So $f(0) = 0$ is a local minima.

(C)

$$f'(x) = (x^2)'F'(x^2) = \frac{2x}{1+(x^2)^4},$$
$$f''(x) = \frac{2\cdot(1+x^8) - 2x\cdot 8x^7}{(1+x^8)^2} = \frac{2}{(1+x^8)^2}(1-7x^8).$$

So

$$\begin{cases} f''(x) > 0 & \Leftrightarrow\ 1-7x^8 > 0\ \Leftrightarrow\ x\in[-c,c], \\ f''(x) < 0 & \Leftrightarrow\ x\in(-\infty,-c)\cup(c,\infty) \end{cases}$$

where $c = 7^{-1/8}$. Hence $f$ is concave up on $[-c,c]$, and $f$ is concave down on $(-\infty,-c]\cup[c,\infty)$.

(D) From (C), we know that $f$ has two inflection points $(\pm c, f(c))$.

`Ans: AC`

13.

(A) Let $f(x) = x^2$ and $g(x) = 2x^2$.
Then $0 < f(x) < g(x)\ \forall\ x \neq 0$. But $L = 0^2 = 0 = 2\cdot 0^2 = M$. (In fact, $L \leq M$.)

(B) Since $0 \leq ||f(x)| - |L|| \leq |f(x) - L|$ and $\lim_{x\to 0} f(x) - L = L - L = 0$, by squeeze theorem, $\lim_{x\to 0} |f(x)| - |L| = 0$, $\lim_{x\to 0} |f(x)| = |L|$.

(C) Let $f(x) = 0 \ \forall\ x \in \mathbb{R}$, and $g(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$

Then $L = \lim_{x\to 0} 0 = 0$.

$$\lim_{x\to 0} g\left(f(x)\right) = \lim_{x\to 0} g\left(0\right) = \lim_{x\to 0} 1 = 1,$$

$$M = \lim_{x\to 0} g(x) = \lim_{x\to 0} 0 = 0 \neq 1.$$

(D)

$$L = \lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = M.$$

`Ans: BD`

14.

(A) The surface area $S =$

$$\int 2\pi y\, dS = \int 2\pi y \sqrt{1 + (y')^2}\, dx = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x}\, dx.$$

(B) Let $f(x) = \cos x \sqrt{1 + \sin^2 x}$. Since

$$\begin{cases} \sin(\pi - x) & = \sin x, \\ \cos(\pi - x) & = -\cos x \end{cases}$$

Thus $f(\pi - x) = -f(x)$,

$$\begin{aligned} I &:= 2\pi \int_0^\pi \cos x \sqrt{1 + \sin^2 x}\, dx \\ &= 2\pi \left( \int_0^{\pi/2} f(x)\, dx + \int_{\pi/2}^{\pi} f(x)\, dx \right) \\ &= 2\pi \left( \int_0^{\pi/2} f(x)\, dx + \int_{\pi/2}^{\pi} -f(\pi - x)\, dx \right) \\ &= 2\pi \left( \int_0^{\pi/2} f(x)\, dx + \int_{\pi/2}^{0} f(x)\, dx \right) = 0. \end{aligned}$$

But the surface area should be positive.

(C) The surface area $S =$

$$\int 2\pi y\, dS = \int 2\pi y \sqrt{1 + (y')^2}\, dx = 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x}\, dx.$$

Let $\tan\theta = \cos x$, then $\sec^2\theta\, d\theta = (-\sin x)\, dx$,

$$(\tan\theta_\ell, \tan\theta_u) = (\cos 0, \cos\pi) = (1, -1) \Rightarrow (\theta_\ell, \theta_u) = \left(\frac{\pi}{4}, -\frac{\pi}{4}\right).$$

By changing of variable,

$$S = 2\pi \int_{\pi/4}^{-\pi/4} \sqrt{1+\tan^2\theta}(-\sec^2\theta)\, d\theta = 2\pi \cdot 2 \int_0^{\pi/4} \sec^3\theta\, d\theta$$

$$(\sqrt{1+\tan^2\theta}(\sec^2\theta) = \sec^3\theta \text{ is an even function.})$$

$$= 2\pi \cdot 2 \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\Big|_{\theta=0}^{\pi/4}\right)$$

$$= 2\pi\left(\sqrt{2} + \ln(\sqrt{2}+1)\right).$$

(D) See (C).

`Ans: AC`

15.

(A) For $f(x) = \arctan x$, $f'(x) = 1/(1+x^2) \in (0,1]\ \forall\ x \in \mathbb{R}$, $\lim_{x\to\infty} f(x) = \dfrac{\pi}{2}$.

(B) Let $f'(0) = m > 0$. Since $f''(x) > 0\ \forall\ x \in \mathbb{R}$, $f'$ is strictly increasing on $\mathbb{R}$, $f'(p) > f'(0) = m > 0$ for all $p > 0$.

Now for $x > 0$, by mean value theorem, there is a $c_x \in (0, x)$ such that

$$f(x) - f(0) = f'(c_x)(x - 0), \quad f'(c_x) > f'(0) = m > 0.$$

So

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(c_x)\cdot x + f(0) \geq \lim_{x\to\infty} mx + f(0) = \infty.$$

(C) Let $f(x) = -e^{-x}$. Then $f'(x) = e^{-x} > 0$, $f''(x) = -e^{-x} < 0$. $\lim_{x\to\infty} f(x) = -0 = 0$.

(D) Let $f'(0) = m > 0$. Since $f''(x) < 0\ \forall\ x \in \mathbb{R}$, $f'$ is strictly decreasing on $\mathbb{R}$, $f'(d) > f'(0) = m > 0$ for all $d < 0$.

Now for $x < 0$, by mean value theorem, there is a $c_x \in (x, 0)$ such that

$$f(x) - f(0) = f'(c_x)(x - 0), \quad f'(c_x) > f'(0) = m > 0.$$

So

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} f'(c_x)\cdot x + f(0) \leq \lim_{x\to-\infty} mx + f(0) = -\infty.$$

Thus for all $M \in \mathbb{R}$, there must exists a real number $x_M$ such that $f(x_M) < M$.

`Ans: BD`

16. Notice that $f(x) = \int_0^{\sqrt{x}} e^{(\sqrt{x}t)^2} \sqrt{x}\, dt$. Set $u = \sqrt{x}t$.
Then $du = \sqrt{x}\, dt$. By changing of variable, $f(x) = \int_0^x e^{u^2}\, du$.
By fundamental theorem of calculus, $f'(x) = e^{x^2}$.

`Ans:` $e^{x^2}$

17.

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} 1 + \sin\frac{1}{x} = 1 + \sin 0 = 1.$$

Now,

$$\begin{aligned} \lim_{x\to\pm\infty} f(x) - 1\cdot x &= \lim_{x\to\pm\infty} x\sin\frac{1}{x} = \lim_{x\to\pm\infty} \frac{\sin(1/x)}{1/x} \\ &= \lim_{t\to 0} \frac{\sin t}{t} \quad (t := \frac{1}{x}, \lim_{x\to\pm\infty} \frac{1}{x} = \lim_{t\to 0} t = 0) = 1. \end{aligned}$$

So $y = 1\cdot x + 1 = x + 1$ is the slant asymptote.

`Ans:` $y = x + 1$

18. $\begin{cases} x &= y^2 + 3 \\ y &\geq 0 \end{cases} \Leftrightarrow \begin{cases} y &= \sqrt{x-3} \\ x &\geq 3 \end{cases}.$

To find the point of intersection, $\begin{cases} x &= y^2 + 3 \\ y &= \dfrac{\sqrt{x}}{2} \end{cases} \Leftrightarrow (x, y) = (4, 1).$

The area =

$$\begin{aligned} &\int_0^4 \frac{\sqrt{x}}{2}\, dx - \int_3^4 \sqrt{x-3}\, dx \\ &= \left( \frac{1}{3}x^{3/2}\Big|_{x=0}^4 \right) - \left( \frac{2}{3}(x-3)^{3/2}\Big|_{x=3}^4 \right) \\ &= \frac{8}{3} - \frac{2}{3} = 2. \end{aligned}$$

`Ans:` 2

19.

$$\begin{aligned} f'(x) &= \frac{2(x^2+3)^2 - 2x\cdot 2(x^2+3)\cdot 2x}{(x^2+3)^2} \\ &= (-6)\cdot \frac{x^2-1}{x^2+3} = 0 \Leftrightarrow x = \pm 1. \end{aligned}$$

Thus $\min_{x\in\mathbb{R}} f(x) = \min\{f(\pm 1)\} = -\frac{1}{8}$.

`Ans:` $-\frac{1}{8}$

20. Let

$$f(x) = \frac{x}{x^2+1} - \frac{a}{2x+1} = \frac{(2-a)x^2 + x - a}{(x^2+1)(2x+1)}.$$

**Case I**: $2 - a > 0$.

For $x \geq 2 > a$,

$$f(x) > \frac{(2-a)x^2 + x - a}{(x^2+x^2)(2x+x)} > \frac{(2-a)x^2}{6x^3} = \frac{2-a}{6}\frac{1}{x} > 0.$$

Since $\int_2^\infty \frac{1}{x}\,dx$ diverges, by comparison test, $\int_2^\infty f(x)\,dx$ diverges.

**Case II**: $2 - a < 0$.

For $x > 1$ with $x > \frac{2}{a-2} > 0$, $\left(\frac{a-2}{2}\right) x > 1$, and thus

$$-f(x) > \frac{(a-2)x^2 - x + a}{(x^2+x^2)(2x+x)}$$

$$> \frac{1}{6x^3}\left\{\left(\frac{a-2}{2}\right)x^2 + x\left[\left(\frac{a-2}{2}\right)x - 1\right]\right\}$$

$$> \frac{1}{6x^3}\cdot\left(\frac{a-2}{2}\right)x^2 = \frac{a-2}{12}\frac{1}{x} > 0.$$

Similar to the case that $2 - a > 0$, $\int_1^\infty -f(x)\,dx$ diverges, $\int_0^\infty f(x)\,dx$ diverges.

**Case III**: $a = 2$.

For $x > a$,

$$0 < f(x) = \frac{x-a}{(x^2+1)(2x+1)} < \frac{x}{x^2\cdot(2x)} = \frac{1}{2x^2}.$$

Since $\int_a^\infty \frac{1}{2x^2}\,dx = \frac{1}{2a}$, by comparison test, $\int_a^\infty f(x)\,dx$ converges, $\int_0^\infty f(x)\,dx$ converges.

`Ans:` 2