SUGGESTED SOLUTION FOR COMPREHENSIVE ASSESSMENT OF CALCULUS (II) IN SUMMER, 2018 (106-2)

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ABSTRACT. This text is a document of suggested solution for comprehensive assessment of Calculus (II), which is held on June 13th, 2018. It happened during the second semester on 106th academical year.

Question 1. [10-4 ★★] Observe that the region is symmetric about the x-axis. Notice that $r = 3\cos\theta \geq 0 \Leftrightarrow -\dfrac{\pi}{2} + 2n\pi \leq \theta \leq \dfrac{\pi}{2} + 2n\pi$ for all $n \in \mathbb{Z}$; $r = 1 + \cos\theta \geq 0 \Leftrightarrow \theta \in \mathbb{R}$.

Now, if these curves intersect at points which lie above x-axis, then they intersect at $(r\cos\theta, r\sin\theta)$ where $3\cos\theta = 1 + \cos\theta$, and $0 \leq \theta < \pi$. So $\theta = \dfrac{\pi}{3}$. Notice that $3\cos\theta \geq 1 + \cos\theta \Leftrightarrow 0 \leq \theta \leq \dfrac{\pi}{3}$.

Hence the area A is equal to

$$2\left(\int_0^{\pi/3} \frac{1}{2}(3\cos\theta)^2\ d\theta - \int_0^{\pi/3} \frac{1}{2}(1+\cos\theta)^2\ d\theta\right) = \int_0^{\pi/3} 8\cos^2\theta - 2\cos\theta - 1\ d\theta$$

$$= \int_0^{\pi/3} 4\cos 2\theta - 2\cos\theta + 3\ d\theta = 2\sin 2\theta - 2\sin\theta + 3\theta\Big|_{\theta=0}^{\pi/3} = \pi.$$

Notice that θ between 0 and $\dfrac{\pi}{3}$ of integrations mean the top half of the region.

`Ans: (B).`

Question 2. [13-2 ★] $\gamma(t_0) = (1, 0, 1) \Leftrightarrow t_0 = 0$.

$$\gamma'(t)|_{t=0} = e^{-t}(-\cos t - \sin t)\ \mathbf{i}\ + e^{-t}(-\sin t + \cos t)\ \mathbf{j}\ - e^{-t}\ \mathbf{k}\ \Big|_{t=0} = -\ \mathbf{i}\ +\ \mathbf{j}\ -\ \mathbf{k}\ = (-1, 1, -1).$$

Thus the equation of the tangent line is

$$L : \begin{cases} x = & 1 - t, \\ y = & \quad t,\ t \in \mathbb{R}. \\ z = & 1 - t, \end{cases}$$

`Ans: (B).`

Question 3. [11-8 ★]

$$\sum_{n=0}^{\infty} 3^n x^{2n} = \sum_{n=0}^{\infty} \left(3x^2\right)^n \text{ converges} \Leftrightarrow \left|3x^2\right| < 1 \Leftrightarrow |x| < \frac{1}{\sqrt{3}}.$$

`Ans: (C).`

Question 4. [11-3 ★★] `(A)` Let $S_n = \sum_{k=1}^{n} a_k$. So $\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} a_k := s \in \mathbb{R}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = s - s = 0.$$

`(B)` $||a_n| - 0| = |a_n| = |a_n - 0|$. By the $\varepsilon - \delta$ definition of limit, $\lim_{n\to\infty} |a_n| = 0$.

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(C) $2 \neq \frac{\pi}{2} + 2n\pi$ for all $n \in \mathbb{Z}$, $\sin 2 < 1$. By p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^{\sin 2}}$ diverges.

(D) Let $a_n = b_n = \left(\frac{1}{2}\right)^n$. Then $A = B = \frac{1/2}{1-1/2} = 1$. But

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3} \neq 1 = AB.$$

Ans: (D).

Question 5. [14-6 ★★★] Let $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$$D_u(0,0) = \lim_{h\to\infty} \frac{f((0,0)+hu) - f(0,0)}{h} = \lim_{h\to\infty} \frac{f\left(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right) - f(0,0)}{h}$$
$$= \lim_{h\to\infty} \frac{1}{h} \left(\frac{\left(\frac{h}{\sqrt{2}}\right)^2 \left(\frac{h}{\sqrt{2}}\right)^3}{\left(\frac{h}{\sqrt{2}}\right)^4 + \left(\frac{h}{\sqrt{2}}\right)^4} \right) = \lim_{h\to\infty} \frac{1}{h} \left(\frac{h^5 \left(\frac{1}{\sqrt{2}}\right)^5}{h^4 \cdot 2 \cdot \left(\frac{1}{\sqrt{2}}\right)^4} \right) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Ans: (B).

Question 6. [14-8 ★] [Way I] Let $f(x,y) = x^2 + y^2$ and $g(x,y) = x^2 - 2x + y^2 - 4y$. If (a,b) reaches the maximum of f, then there exists a $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \nabla f(a,b) &= \lambda \nabla g(a,b), \\ g(a,b) &= 0 \end{cases} \Leftrightarrow \begin{cases} 2a = \lambda(2a-2), & (1) \\ 2b = \lambda(2b-4), & (2) \\ a^2 - 2a + b^2 - 4b = 0. & (3) \end{cases}$$

Eq. (1), (2) implies that $\begin{cases} (\lambda-1)a &= \lambda, \\ (\lambda-1)b &= 2\lambda. \end{cases}$ Hence $(\lambda-1)\cdot 2a = 2\lambda = (\lambda-1)b$. If $\lambda = 1$, then $2a = 2a-2$ by (1), contradiction. So $\lambda - 1 \neq 0$, $2a = b$. Thus by (3), $a^2 - 2a + (2a)^2 - 4\cdot(2a) = 5a^2 - 10a = 0$.

a	(a,b)	$f(a,b)$
0	$(0,0)$	0
2	$(2,4)$	20

[Way II] Notice that $x^2 + y^2$ is the square of distance of points (x,y) and the origin. The constraint $(x-1)^2 + (y-2)^2 = 5$ is a circle centered at $(1,2)$ with radius $\sqrt{5}$. Hence the maxima of x^2+y^2 is $\left(\|(0,0)-(1,2)\| + \sqrt{5}\right)^2 = (2\sqrt{5})^2 = 20$.

Ans: (B).

Question 7. [14-2 ★★] Notice that $\lim_{t\to1}(t,1) = \lim_{t\to1}(t,t) = (1,1)$. But

$$\begin{cases} \lim_{t\to1} \frac{(\ln t)(\ln 1)}{(\ln t)^2 + (\ln 1)^2} = \lim_{t\to1} \frac{0}{(\ln t)^2 + (\ln t)^2} = 0, \\ \lim_{t\to1} \frac{(\ln t)(\ln t)}{(\ln t)^2 + (\ln 1)^2} = \lim_{t\to1} \frac{(\ln t)^2}{2(\ln t)^2} = \frac{1}{2}. \end{cases}$$

Ans: (D).

Question 8. [15-5 ★★] Let $\Omega = \{(x, y) \in \mathbb{R} \mid x^2+y^2 \leq 1 \text{ and } z = xy \geq 0\}$. Then for $x = r\cos\theta$, $y = r\sin\theta$, we have

$$x^2 + y^2 \leq 1 \text{ and } z = xy \geq 0 \Leftrightarrow r^2 \leq 1, r^2\cos\theta\sin\theta \geq 0 \Leftrightarrow 0 \leq r \leq 1, \sin 2\theta \geq 0$$
$$\Leftrightarrow 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2} \text{ or } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.$$

Thus the area of the surface is

$$A = \iint_\Omega \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy = \iint_\Omega \sqrt{1 + y^2 + x^2}\, dxdy$$
$$= \int_0^{\pi/2}\int_0^1 \sqrt{1+r^2}\, r\, drd\theta + \int_0^{3\pi/2}\int_0^1 \sqrt{1+r^2}\, r\, drd\theta \text{ (change of variable by polar coordiante)}$$
$$= \pi\int_0^1 \sqrt{1+r^2}\, r\, drd\theta = \pi \cdot \frac{1}{3}(1+r^2)^{3/2}\Big|_{r=0}^{1} = \frac{\pi}{3}\left(2\sqrt{2}-1\right).$$

Ans: (C).

Question 9. [15-2 ★★] (A)

$$\int_0^1\int_{x^2-1}^{x-1} dydx = \int_0^1 (x-1) - (x^2-1)\, dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_{x=0}^{1} = \frac{1}{6}.$$

(B)

$$\int_0^{-1}\int_{\sqrt{y+1}}^{y+1} dxdy = \int_0^{-1} y + 1 - \sqrt{y+1}\, dy = \frac{1}{2}y^2 + y - \frac{2}{3}(y+1)^{3/2}\Big|_{y=0}^{-1} = \left(\frac{1}{2}-1\right) - \left(-\frac{2}{3}\right) = \frac{1}{6}.$$

(C)

$$\int_0^1\int_{y+1}^{\sqrt{y+1}} dxdy = \int_0^1 \sqrt{y+1} - y - 1\, dy = \frac{2}{3}(y+1)^{3/2} - \frac{1}{2}y^2 - y\Big|_{y=0}^{1} = \left(\frac{2}{3}\cdot 2\sqrt{2} - \frac{1}{2} - 1\right) - \frac{2}{3}.$$

Ans: (C).

Question 10. [14-4 ★★] Let $f(x, y, z) = xyz + 2yz + x^2z^2 - 1$ and $g(x, y, z) = x^2y^2z^2 + yz - x^2z + 1$. Then

$$\nabla f(x, y, z) = (yz + 2xz^2, xz + 2z, xy + 2y + 2x^2z),$$
$$\nabla g(x, y, z) = (2xy^2z^2 - 2xz, 2x^2yz^2 + z, 2x^2y^2z + y - x^2).$$

Then the direction of the line $(a, 2, b)$ is parallel to $\nabla f(1, 0, 1) \times \nabla g(1, 0, 1) = (2, 3, 2) \times (-2, 1, -1) = (-5, -2, 8)$. So $(a, 2, b) = (5, 2, -8)$, $a + b = -3$.

Ans: (A).

Question 11. [11-6 ★★] (A) Since $\left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right| > \frac{1}{n+1} \geq \frac{1}{2n} > 0$ and $\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges by p-series test, so $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ diverges by comparison test.

On the other hand, $\frac{1}{\sqrt{n(n+1)}} \downarrow 0$ as $n \to \infty$. By alternating test and the paragraph above, this alternating series converges conditionally.

(B) Let $a_n = \left|\frac{(-1)^{n+1}2^n}{(2n+1)!}\right| = \frac{2^n}{(2n+1)!}$. Then $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2}{2n+3} = 0 < 1$. Hence the series converges absolutely by ratio test.

(C) Set $f(x) = \sqrt[3]{x} - \ln x$. Then $f'(x) = \frac{1}{3}x^{-2/3} - \frac{1}{x} = \frac{1}{3x}(\sqrt[3]{x} - 3) > 0 \Leftrightarrow \sqrt[3]{x} - 3 > 0 \Leftrightarrow x > 27$, f is strictly increasing on $(27, \infty)$. While $x > e^6 > 2^6 = 64 > 27$, $f(x) > f(e^6) = e^2 - 6 > \left(1 + \frac{1}{1!} + \frac{1}{2!}\right)^2 - 6 = 6.25 - 6 > 0$.

Therefore, when $n > e^6$, $\sqrt[3]{n} - \ln n > 0$, $\left|\frac{(-1)^{n+1}(\ln n)^2}{n^2}\right| < \frac{(\sqrt[3]{n})^2}{n^2} = \frac{1}{n^{4/3}}$. $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ converges by p-series test. So this alternating series converges absolutely by comparison test.

(D) $\left|\frac{\arctan n}{n^2}\right| < \frac{\pi}{2n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ converges by p-series test. So the series converges absolutely by comparison test.

Ans: (B)(C)(D).

Question 12. [15-1 ★★★] (A) This is fine, for any integrable function.

(B) First, $F(x, y) = \int_0^x \int_0^y f(s, t)\ dtds$ by Fubini's Theorem. Set $h_y(s) = \int_0^y f(s, t)\ dt$ for all $(s, y) \in \mathbb{R}^2$. Then for any $y \in \mathbb{R}$, h_y is continuous on $\mathbb{R}$. By Fundamental Theorem of Calculus, $F_x(x, y) = \frac{\partial}{\partial x} \int_0^x h_y(s)\ ds = h_y(x) = \int_0^y f(x, t)\ dt$. Similarly, $F_y(x, y) = \frac{\partial}{\partial y} \int_0^y \int_0^x f(s, t)\ dsdt = \int_0^x f(s, y)\ ds$. Hence F_x, F_y are both continuous on $\mathbb{R}^2$, F is differentiable on $\mathbb{R}^2$.

(C) By (B), $F_{xy}(x, y) = \frac{\partial}{\partial y} \int_0^y f(x, t)\ dt = f(x, y)$. Similarly, $F_{yx}(x, y) = \frac{\partial}{\partial x} \int_0^x f(s, y)\ ds = f(x, y)$.

(D) By (B), for $x > 0$,

$$F_x(x, x) + F_y(x, x) = \int_0^x f(x, t)\ dt + \int_0^x f(s, x)\ ds$$
$$= \int_0^x f(x, u) + f(u, x)\ du = \int_0^x f(x, u) - f(x, u)\ du = 0.$$

Ans: (A)(B)(D).

Question 13. [13-2 ★] $6 - x - x^2 = -(x + 3)(x - 2) \geq 0 \Leftrightarrow -3 \leq x \leq 2$. Thus

$$\max_{\alpha \leq \beta} \int_\alpha^\beta (6 - x - x^2)\ dx = \int_{-3}^2 (6 - x - x^2)\ dx.$$

Ans: (A)(D).

Question 14. [14-8 ★★] Let $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = a^2b^2 - (a - b)^2 = (a^2 - 1)(b^2 - 1) + 2ab - 1 = 2ab - 1$ for $(a, b) = (\pm 1, \pm 1)$. Then

(a, b)	$\pm(1, 1)$	$\pm(1, -1)$
$D(a, d)$	1	-3
$f_{xx}(a, b)$	1	-3
extreme value type	min.	saddle point

Ans: (A)(D).

Question 15. [15-6 ★★★]

$$\begin{cases} 0 & \leq z \leq 1 - y, \\ \sqrt{x} & \leq y \leq 1, \\ 0 & \leq x \leq 1 \end{cases} \Leftrightarrow \begin{cases} \sqrt{x} & \leq y \leq 1 - z, \\ 0 & \leq x \leq y^2, \\ 0 & \leq y \leq 1, \\ 0 & \leq z \leq 1 - y \leq 1. \end{cases}$$

Hence

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\ dzdydx = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\ dzdxdy = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\ dxdzdy$$

$$= \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\ dxdydz = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\ dydxdz = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\ dydzdx.$$

The choice (A) is just changing the dummy variable.

Ans: (A)(B)(C).

Question 16. [10-3 ★★] Notice that $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\frac{1}{\theta^2}\sin\theta + \frac{1}{\theta}\cos\theta}{-\frac{1}{\theta^2}\cos\theta - \frac{1}{\theta}\sin\theta} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

if $\theta \neq 0$ and $-\cos\theta - \theta\sin\theta \neq 0$. Hence

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \left.\frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}\right|_{\theta=\pi} = -\pi.$$

Ans: $-\pi$.

Question 17. [11-8 ★] For $x \neq 0$, by ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)\cdot 3^{n+1}} x^{n+1}}{\frac{(-1)^n}{n\cdot 3^n} x^n} \right| = \lim_{n\to\infty} \frac{n}{3(n+1)}|x| = \frac{|x|}{3} < 1 \Leftrightarrow x \in (-3,3).$$

While $x = 3$, the power series is equal to $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$, which converges conditionally by alternating test and p-series test.

While $x = -3$, the power series is equal to $\sum_{n=1}^{\infty} \dfrac{1}{n}$, which diverges by p-series test.

Ans: $(-3, 3]$.

Question 18. [15-9 ★★★★]

line segment	$\overline{(1,0),(2,0)}$	$\overline{(2,0),(0,-2)}$	$\overline{(0,-2),(0,-1)}$	$\overline{(0,-1),(1,0)}$
line equation in (x,y)	$y = 0$	$x - y = 2$	$x = 0$	$x - y = 1$
line equation in (u,v)	$u = v$	$v = 2$	$u = -v$	$v = 1$

The area of $T(E)$ is

$$\iint_{T(E)} dudv = \int_1^2 \int_{-v}^{v} dudv = \int_1^2 [v - (-v)]\ dv = \int_1^2 2v\ dv = v^2\Big|_{v=1}^{2} = 3.$$

Ans: 3.

Question 19. [15-1 ★★★★] The amount of (i,j) in double summation is $n \cdot n^2 = n^3$, thus $\Delta x \Delta y = \dfrac{c}{n^3}$ for some constant c. Now, $n^{-2}\dfrac{1}{\sqrt{n^2+ni+j}} = \dfrac{1}{n^3}\dfrac{1}{\sqrt{1+(\frac{i}{n})+(\frac{j}{n^2})}}$,

	$i=1$	$i=n$
$\frac{i}{n}$	$\frac{1}{n}$	1

	$j=1$	$i=n^2$
$\frac{j}{n^2}$	$\frac{1}{n^2}$	1

, $\dfrac{1}{n}$ and $\dfrac{1}{n^2} \to 0$ as $n \to \infty$.

Hence

$$\lim_{n\to\infty} n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{n^2+ni+j}} = \lim_{n\to\infty}\frac{1}{n}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n^2}\frac{1}{\sqrt{1+(\frac{i}{n})+(\frac{j}{n^2})}}$$

$$= \iint_{0\le x,y\le 1}\frac{dxdy}{\sqrt{1+x+y}} = \int_0^1\int_0^1 \frac{1}{\sqrt{1+x+y}}\ dxdy = \int_0^1 \left(2\sqrt{1+x+y}\Big|_{x=0}^{1}\right)\ dy$$

$$= 2\int_0^1 \left(\sqrt{2+y}-\sqrt{1+y}\right)\ dy = 2\left[\frac{2}{3}\left((2+y)^{3/2}-(1+y)^{3/2}\right)\right]$$

$$= \frac{4}{3}\left[\left(3\sqrt{3}-2\sqrt{2}\right)-\left(2\sqrt{2}-1\right)\right] = \frac{4}{3}\left(3\sqrt{3}-4\sqrt{2}+1\right).$$

`Ans:` $\frac{4}{3}\left(3\sqrt{3}-4\sqrt{2}+1\right)$.

Question 20. [13-2 ★] $y = 0 \le 1-x^2 \Leftrightarrow -1 \le x \le 1$. The volume is equal to

$$\int_0^1\int_{-1}^1\int_0^{1-x^2}\ dydxdz = (1-0)\cdot\int_{-1}^1 \left(1-x^2\right)\ dx = x-\frac{1}{3}x^3\Big|_{x=-1}^{1} = \frac{4}{3}.$$

`Ans:` $\frac{4}{3}$.

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