SUGGESTED SOLUTION FOR COMPREHENSIVE ASSESSMENT OF CALCULUS (I) IN WINTER, 2019 (107-1)

TÂN UÍ-TSIÅT

Abstract. This text is a document of suggested solution for comprehensive assessment of Calculus (I), which is held on January 9th, 2019. It happened during the first semester on 107th academical year.

Question 1. [06.5-S-★] For $x > 0$, let $f(x)$ be the **average** value of e^{-t} on $[0, x]$. How many **critical numbers** on $(0, \infty)$ does f have?

`(A) 0;  (B) 1;  (C) 2;  (D) 3.`

Solution 1. $f(x) = \dfrac{1}{x}\displaystyle\int_0^x e^{-t}\,dt = \dfrac{1}{x}\left(-e^{-t}\Big|_{t=0}^{x}\right) = \dfrac{1-e^{-x}}{x}$ for $x > 0$.

$$f'(x) = \frac{(-1)(-e^{-x})\cdot x - (1-e^{-x})\cdot 1}{x^2} = \frac{(x+1)e^{-x}-1}{x^2} = 0 \Leftrightarrow (x+1)e^{-x} - 1 = 0 \Leftrightarrow x+1 = e^x.$$

Set $g(x) = e^x - (x+1)$. Then $g(0) = 0$ and $g'(x) = e^x - 1 > 0$ for $x > 0$. By Mean Value Theorem, for any $x > 0$, there exists a $c \in (0, x)$ such that $g(x) - g(0) = g'(c)(x-0) > 0$. Thus $e^x > x+1$ when $x > 0$, $f'(x) < 0$ for all $x > 0$.

`Ans: (A).`

Question 2. [03.6-S-★] Let $f(x) = \left(1 + \dfrac{a}{x}\right)^{1/x}$, where $a \neq 0$. Then $f'(a) =$

(A) $\dfrac{2^{1/a}}{2a}$; (B) $-\dfrac{2^{1/a}}{2a^2}$; (C) $-\dfrac{2^{1/a}}{a^2}\left(\ln 2 - \dfrac{1}{2}\right)$; (D) $-\dfrac{2^{1/a}}{a^2}\left(\ln 2 + \dfrac{1}{2}\right)$.

Solution 2.

$$\begin{aligned}
\ln f(x) &= \frac{1}{x}\ln\left(\frac{x+a}{x}\right) = \frac{1}{x}[\ln(x+a) - \ln x] \\
\Rightarrow \frac{d}{dx}[f(x)] &= \frac{f'(x)}{f(x)} = \frac{-1}{x^2}[\ln(x+a) - \ln x] + \frac{1}{x}\left[\frac{1}{x+a} - \frac{1}{x}\right] \\
\Rightarrow f'(a) &= f(a)\left\{\left(\frac{-1}{a^2}\right)[\ln(2a) - \ln a] + \frac{1}{a}\left(\frac{1}{2a} - \frac{1}{a}\right)\right\} = 2^{1/a}\left(\frac{-1}{a^2}\right)\left(\ln 2 + \frac{1}{2}\right).
\end{aligned}$$

`Ans: (D).`

Question 3. [05.3-S-★] Let $f(x) = \displaystyle\int_{2x}^{3-x} e^{t^2}\,dt$. Then $(f^{-1})'(0) =$

(A) $\dfrac{-1}{3e^4}$; (B) $\dfrac{1}{e^9 - 1}$; (C) $\dfrac{-1}{e^9+2}$; (D) $\dfrac{1}{e^4}$.

Solution 3. Notice that $e^{t^2} > 0$ for all $t \in \mathbb{R}$. So $f(x) = 0 \Leftrightarrow 3 - x = 2x \Leftrightarrow x = 1$; $f^{-1}(0) = 1$. By Inverse Function Theorem, $(f^{-1})'(0) = \dfrac{1}{f'(1)}$. Set $g(x) = \displaystyle\int_0^x e^{-t^2}\,dt$ for all $x \in \mathbb{R}$. Then $f(x) = g(3-x) - g(2x)$.

Date: July 19, 2019.

Key words and phrases. Limit, derivative, Riemann integral, volume, parametric equation.

Thanks to Ellie Sung.

By Fundamental Theorem of Calculus, $g'(x) = e^{-x^2}$. From Chain Rule, we obtain that

$$f'(x) = e^{(3-x)^2} \cdot (3-x)' - e^{(2x)^2} \cdot (2x)' = -e^{(3-x)^2} - 2e^{(2x)^2},$$

Therefore, $f'(1) = -3e^4$, $(f^{-1})'(0) = -\dfrac{1}{3e^4}$.

`Ans: (A).`

Question 4. [07.4-S-★★★★] The **value** of $\displaystyle\int_1^{\sqrt{3}} \frac{x^4 - x^3 + 2x^2 + 1}{x(x^2+1)^2}\,dx$ is

(A) $\ln\sqrt{3} - \frac{\pi}{24} + \frac{\sqrt{3}-2}{4}$; (B) $\ln\sqrt{3} - \frac{\pi}{12} + \frac{\sqrt{3}-2}{4}$; (C) $\ln\sqrt{3} - \frac{\pi}{24} + \frac{\sqrt{3}-2}{8}$; (D) $\ln\sqrt{3} - \frac{\pi}{12} + \frac{\sqrt{3}-2}{8}$.

Solution 4. $x^4 - x^3 + 2x^2 + 1 = (x^2+1)^2 - x^3 = (x^2+1)^2 - x(x^2+1) + x$. Thus

$$\frac{x^4 - x^3 + 2x^2 + 1}{x(x^2+1)^2} = \frac{(x^2+1)^2 - x(x^2+1) + x}{x(x^2+1)^2} = \frac{1}{x} - \frac{1}{x^2+1} + \frac{1}{(x^2+1)^2}.$$

So we will separate it into three parts. Clearly,

$$\int_1^{\sqrt{3}} \frac{1}{x}\,dx = \ln x\Big|_{x=1}^{\sqrt{3}} = \ln\sqrt{3}, \text{ and } \int_1^{\sqrt{3}} \frac{-1}{1+x^2}\,dx = -\tan^{-1} x\Big|_{x=1}^{\sqrt{3}} = (-1)\cdot\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = -\frac{\pi}{12}.$$

To solve the third part, we apply changing of variable, set $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, $(\tan\theta_1, \tan\theta_2) = (1, \sqrt{3}) \Leftrightarrow (\theta_1, \theta_2) = (\pi/4, \pi/3)$.

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{1}{(x^2+1)^2}\,dx &= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{2\cos^2\theta - 1}{2} + \frac{1}{2}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\cos 2\theta}{2} + \frac{1}{2}\,d\theta = \frac{\sin(2\theta)}{4} + \frac{\theta}{2}\Big|_{\theta=\pi/4}^{\pi/3} \\
&= \left(\frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) - \left(\frac{1}{4}\cdot 1 + \frac{\pi}{8}\right) = \frac{\sqrt{3}}{8} + \frac{\pi}{24} - \frac{1}{4}.
\end{aligned}$$

Thus the value is $\ln\sqrt{3} + \left(-\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{8} + \frac{\pi}{24} - \frac{1}{4}\right) = \ln\sqrt{3} - \frac{\pi}{24} + \frac{\sqrt{3}-2}{8}$.

`Ans: (C).`

Question 5. [06.3-S-★] The region bounded by curves $y = e^{-x}$, $y = 0$, $x = 0$ and $x = 1$ is rotated about the y-axis. Then the **volume** of the resulting solid of revolution is

(A) $\frac{\pi}{2}(1 - e^{-2})$; (B) $2\pi(1 - 2e^{-1})$; (C) $2\pi(1 - e^{-1})$; (D) $\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$.

Solution 5. By shell method,

$$V = \int_0^1 2\pi x f(x)\,dx = \int_0^1 2\pi x e^{-x}\,dx = 2\pi(-x-1)e^{-x}\Big|_{x=0}^{1} = 2\pi[-2e^{-1} - (-1)].$$

`Ans: (B).`

Question 6. [05.3-S-★★★★] The **limit** $\displaystyle\lim_{x\to 0} \frac{\int_0^x \left(\int_0^{\sin t} \sqrt{1+u^2}\,du\right)dt}{\tan^2 x} =$

(A) 0; (B) $\frac{1}{2}$; (C) 1; (D) Does not exist.

Solution 6. Let $f(u) = \sqrt{1+u^2}$ and $g(t) = \displaystyle\int_0^{\sin t} f(u)\,du$. Then both f and g are continuous on $\mathbb{R}$. By Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_0^x g(t)\,dt = g(x) \text{ and } \frac{d}{dt}g(t) = (\sin t)' f(\sin t) = \cos t\sqrt{1+\sin^2 t}.$$

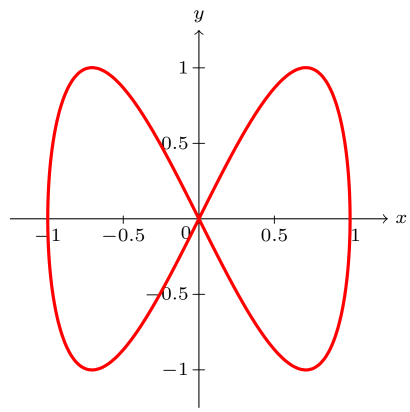
Now, by L'Hopital's rule,

$$\lim_{x\to 0}\frac{\int_0^x \left(\int_0^{\sin t}\sqrt{1+u^2}\,du\right)\,dt}{\tan^2 x} = \lim_{x\to 0}\cos^2 x\frac{\int_0^x g(t)\,dt}{\sin^2 x}$$

$$\overset{L'H}{=} 1^2\cdot\lim_{x\to 0}\frac{g(x)}{2\sin x\cos x} \overset{L'H}{=} \frac{1}{2\cdot 1}\lim_{x\to 0}\frac{\cos x\sqrt{1+\sin^2 x}}{\cos x} = \frac{1}{2}\cdot\sqrt{1+\sin^2 0} = \frac{1}{2}.$$

`Ans: (B).`

Question 7. [10.1-S-★★] Which pair of **parametric equations** represents the graph below?



(A) $\begin{cases} x=\cos\theta, \\ y=\theta+\sin\theta \end{cases}$; (B) $\begin{cases} x=\sin(3\theta), \\ y=\cos(4\theta) \end{cases}$; (C) $\begin{cases} x=\theta-\sin\theta, \\ y=\cos\theta \end{cases}$; (D) $\begin{cases} x=\sin\theta, \\ y=\sin(2\theta) \end{cases}$.

Solution 7. (A)

$$x=0\Leftrightarrow\theta=\frac{\pi}{2}+n\pi,\ n\in\mathbb{Z}\Rightarrow y=\frac{\pi}{2}+n\pi+(-1)^n\neq 0,\ n\in\mathbb{Z}.$$

(B)

$$x=0\Leftrightarrow\theta=\frac{n\pi}{3},\ n\in\mathbb{Z}\Rightarrow y=\cos\frac{4n\pi}{3}=\frac{-1\pm\sqrt{3}}{2}\text{ or }1\neq 0,\ n\in\mathbb{Z}.$$

(C)

$$y=0\Leftrightarrow\theta=\frac{\pi}{2}+n\pi,\ n\in\mathbb{Z}\Rightarrow y=\frac{\pi}{2}+n\pi-(-1)^n\neq 0,\ n\in\mathbb{Z}.$$

(D) $(x,y)=(0,0)\Leftrightarrow\theta=n\pi,\ n\in\mathbb{Z}.$

Extreme	θ	$\phi(\theta)=(x,y)=(\sin\theta,\sin 2\theta)$
$\max x$	$\frac{\pi}{2}+2n\pi,\ n\in\mathbb{Z}$	$(1,0)$
$\min x$	$-\frac{\pi}{2}+2n\pi,\ n\in\mathbb{Z}$	$(-1,0)$
$\max y$	$\frac{\pi}{4}+n\pi,\ n\in\mathbb{Z}$	$\left(\pm\frac{\sqrt{2}}{2},1\right)$
$\min y$	$-\frac{\pi}{4}+n\pi,\ n\in\mathbb{Z}$	$\left(\pm\frac{\sqrt{2}}{2},-1\right)$

`Ans: (D).`

Question 8. [05.1-S-★★★] The greatest integer function $[x]$ is a function from $\mathbb{R}$ to $\mathbb{Z}$ with $x-1<[x]\leq x$. The **value** of $\int_0^2 [x^2]\,dx$ is

(A) $\frac{8}{3}$; (B) 1; (C) $7-\sqrt{2}-\sqrt{3}$; (D) $5-\sqrt{2}-\sqrt{3}$.

Solution 8. Notice that $0 \le x \le 2 \Rightarrow 0 \le x^2 \le 4$. Moreover, for any $x \ge 0$ and $n \in \mathbb{Z}^+$, $n-1 < x^2 \le n \Leftrightarrow \sqrt{n-1} < x \le \sqrt{n}$. Thus

$$\begin{aligned}\int_0^2 [x^2]\ dx &= \int_0^1 [x^2]\ dx + \int_1^{\sqrt{2}} [x^2]\ dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]\ dx + \int_{\sqrt{3}}^{2} [x^2]\ dx \\ &= \int_0^1 0\ dx + \int_1^{\sqrt{2}} 1\ dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\ dx + \int_{\sqrt{3}}^{2} 3\ dx \\ &= 0 + (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5 - \sqrt{2} - \sqrt{3}.\end{aligned}$$

`Ans: (D).`

Question 9. [05.3-S-★★] Let f be a **continuous** function on $\mathbb{R}$ satisfying $f(x) = x^{-5} \int_0^x \left(1 - \cos(t^2)\right)\ dt$ for $x \neq 0$. Then $f(0)$ equals
(A) $\frac{1}{2}$; (B) $\frac{1}{5}$; (C) $\frac{1}{10}$; (D) $\frac{1}{20}$.

Solution 9. Since f is continuous at 0, $\lim_{x\to 0} f(x) = f(0)$. By L'Hopital's rule and Fundamental Theorem of Calculus,

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{\int_0^x 1-\cos(t^2)\ dt}{x^5} \overset{l'H}{=} \lim_{x\to 0} \frac{1-\cos(x^2)}{5x^4} \\ &= \lim_{x\to 0} \frac{1-\left(1-2\sin^2(x^2/2)\right)}{5x^4} = \frac{2}{5}\lim_{x\to 0}\left(\frac{\sin(x^2/2)}{x^2}\right)^2 \\ &= \frac{2}{5}\lim_{x\to 0}\left(\frac{1}{2}\frac{\sin(x^2/2)}{x^2/2}\right)^2 = \frac{2}{5}\lim_{x\to 0}\left(\frac{1}{2}\frac{\sin\theta}{\theta}\right)^2 \qquad \left(\text{Let } \theta = \frac{x^2}{2}.\right) \\ &= \frac{2}{5}\cdot\left(\frac{1}{2}\cdot 1\right)^2 = \frac{1}{10}.\end{aligned}$$

`Ans: (C).`

Question 10. [06.2-S-★★] The base of a solid S is the region enclosed by curves $y = \sec x$, $y = \tan x$, $x = 0$ and $x = \pi/4$. The cross-sections perpendicular to the x-axis are **squares**. Then the **volume** of S is
(A) $4 - 2\sqrt{2} - \pi/2$; (B) $4 - \sqrt{2} - \pi/2$; (C) $4 - 2\sqrt{2} - \pi/4$; (D) $4 - \sqrt{2} - \pi/4$.

Solution 10. The length of edge of square at $x = \alpha$ is $\sec\alpha - \tan\alpha$. So the volume of S is

$$\begin{aligned}V &= \int_0^{\pi/4} (\sec x - \tan x)^2\ dx = \int_0^{\pi/4} \sec^2 x + \tan^2 x - 2\sec x \tan x\ dx \\ &= \int_0^{\pi/4} (2\sec^2 x - 1) - 2\sec x\tan x\ dx = 2\tan x - 2\sec x - x\Big|_{x=0}^{\pi/4} \\ &= \left(2 - 2\sqrt{2} - \frac{\pi}{4}\right) - (0 - 2 - 0) = 4 - 2\sqrt{2} - \frac{\pi}{4}.\end{aligned}$$

`Ans: (C).`

Question 11. [05.2-M-★] Which of the following statements are **true**?
(A) If both $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$ hold, then it follows that $\lim_{x\to 0}(f(x) - g(x)) = 0$.
(B) The equation $x^{10} - 10x^2 + 8 = 0$ has a root in $(0, 10)$.
(C) If $|f|$ is integrable on $[0, 1]$, then so is f.
(D) Every continuous function defined on $\mathbb{R}$ has at most two horizontal asymptotes.

Solution 11. (A) For any $c \in \mathbb{R}$, set $g(x) = f(x) - c$ near 0 and $\lim_{x\to 0} f(x) = \infty$. Then $\lim_{x\to 0}(f(x) - g(x)) = \lim_{x\to 0} c = c$.

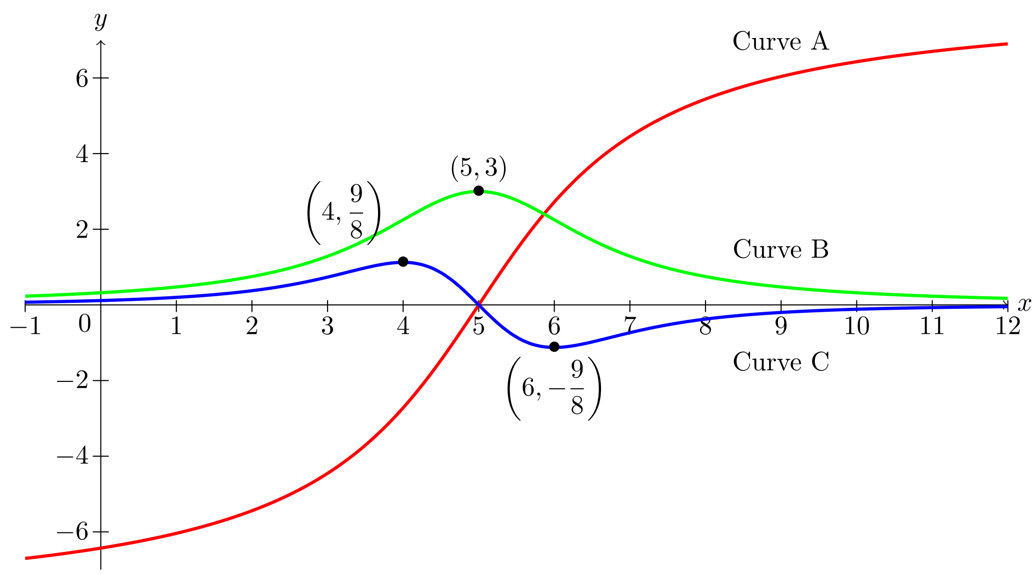
(B) Let $f(x) = x^{10} - 10x^2 + 8$. Then $f(0)f(1) = 8 \cdot (-1) < 0$. By Intermediate Value Theorem, $f(c) = 0$ for some $c \in (0, 1)$.

(C) Set $f : [0, 1] \to \mathbb{R}$ such that $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ -1, & \text{if } x \notin \mathbb{Q}. \end{cases}$ Then f is non-integrable on $[0, 1]$ (since the Riemann sum diverges), but $\int_0^1 |f(x)|\, dx = \int_0^1 1\, dx = (1 - 0) \cdot 1 = 1$.

(D) $y = c$ is a horizontal asymptote of f if and only if $\lim_{x\to\infty} f(x) = c$ or $\lim_{x\to-\infty} f(x) = c$.

Ans: (B)(D).

Question 12. [04.5-M-★★★] The figure below shows graphs of f, f' and f''. Which of the following statements are **true**?



(A) The curve B represents the graph of f.

(B) f attains a local maximum at $x = 5$. (C) f is concave upward on $(0, 5)$.

(D) The largest slope of the graph of f on $[0, 10]$ is happened at $x = 5$.

Solution 12. Suppose that g and h are corresponding to the curve B and C, respectively. Then h has local extreme values at $x = 4$ and $x = 6$, $h'(4) = h'(6) = 0$, h' is not corresponding to any curve in the figure. g has two inflection points, so g'' has two zeros, which is also not corresponding to any curve in the figure. In summary,

Function	f	f'	f''
Curve	A	B	C

In fact, $f(x) = 3\sqrt{3}\tan^{-1}\left(\frac{x-5}{\sqrt{3}}\right)$.

Ans: (C)(D).

Question 13. [07.8-M-★★★] Which ones are **convergent**?

(A) $\int_2^\infty \frac{1}{x(\ln x)^3}\, dx$; (B) $\int_0^{1/2} \frac{1}{x^2(\ln x)^4}\, dx$; (C) $\int_{-1}^1 \sqrt[3]{\frac{\sin x}{x^2}}\, dx$; (D) $\int_0^1 \frac{\sin\sqrt{x}}{x}\, dx$.

Solution 13. (A)

$$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = -\frac{1}{2}\frac{1}{(\ln x)^2}\bigg|_{x=2}^{\infty} = \left(-\frac{1}{2}\right)\left[\frac{1}{\infty^2} - \frac{1}{(\ln 2)^2}\right] = \frac{1}{2(\ln 2)^2}.$$

(B) Since

$$\lim_{x\to 0^+} x(\ln x)^4 = \lim_{x\to 0^+}\frac{(\ln x)^4}{1/x} \overset{L'H}{=} \lim_{x\to 0^+}\frac{4(\ln x)^3/x}{-1/x^2} = \lim_{x\to 0^+}(-4)x(\ln x)^3 = \ldots = \lim_{x\to 0^+} 24x = 0.$$

For some $\delta > 0$, $x(\ln x)^4 < 1$, $\dfrac{1}{(\ln x)^4} > x$ while $0 < x < \delta$. Thus

$$0 < x < \delta \Rightarrow \frac{1}{x^2(\ln x)^4} > \frac{x}{x^2} = \frac{1}{x} > 0,$$

$$\int_0^\delta \frac{1}{x}\,dx = \lim_{t\to 0^+}\left(\ln x\bigg|_{x=t}^{\delta}\right) = \lim_{t\to 0^+}\ln\delta - \ln t = +\infty.$$

By Comparison test, the improper integral diverges.

(C)

$$0 \le \left|\sqrt[3]{\frac{\sin x}{x^2}}\right| \le \frac{1}{|x|^{2/3}} \text{ and}$$

$$\int_{-1}^1 \frac{1}{|x|^{2/3}}\,dx = 2\int_0^1 \frac{1}{x^{2/3}}\,dx = 6x^{1/3}\bigg|_{x=0}^{1} = 6.$$

By Comparison test, the improper integral converges.

(D)

$$0 \le x \le 1 \Rightarrow 0 \le \sqrt{x} \le 1 \le \frac{\pi}{2} \Rightarrow 0 \le \sin\sqrt{x} \le \sqrt{x}.$$

So

$$0 \le \frac{\sin\sqrt{x}}{x} \le \frac{\sqrt{x}}{x}, \text{ and } \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\bigg|_{x=0}^{1} = 2.$$

By Comparison test, the improper integral converges.

Ans: (A)(C)(D).

Question 14. [04.1-M-★★★] Which of the following statements are **true**?

(A) If f is differentiable and increasing on $\mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

(B) If $f(x)$ has a critical point at $x = c$, then $f'(c) = 0$.

(C) If $f(t)$ is differentiable on $(0, \infty)$ and has a limit as $t \to \infty$, then $\lim_{t\to\infty} f'(t) = 0$.

(D) A continuous function on $[a, b]$ attains its absolute maximum.

Solution 14. (A) Clearly, $f(x) = x$ is strictly increasing and differentiable on $\mathbb{R}$, but $f(-1) = -1 < 0$.

(B) $f(x) = |x|$ has a critical point at $x = 0$ since f is not differentiable at $x = 0$. Notice that

$$\lim_{h\to 0^+}\frac{f(0+h) - f(0)}{h} = 1 \ne \lim_{h\to 0^-}\frac{f(0+h) - f(0)}{h} = -1.$$

(C) For $f(t) = \dfrac{\sin(t^2)}{t}$, $t > 0$, we see that $|f(t)| \le \dfrac{1}{t}$ and $\lim_{t\to\infty}\dfrac{1}{t} = 0$, so $\lim_{t\to\infty} f(t) = 0$ by Pinching Theorem. Now,

$$f'(t) = \frac{2t\cos(t^2)\cdot t - \sin(t^2)}{t^2},$$

$$\lim_{n\to\infty}\sqrt{2n\pi} = \infty \text{ and } \lim_{n\to\infty} f'(\sqrt{2n\pi}) = \lim_{n\to\infty} 2\cos(2n\pi) = 2 \ne 0;$$

$$\lim_{n\to\infty}\sqrt{2n\pi + \frac{\pi}{2}} = \infty \text{ and } \lim_{n\to\infty} f'\left(\sqrt{2n\pi + \frac{\pi}{2}}\right) = \lim_{n\to\infty}\frac{-1}{2n\pi + \pi/2} = 0.$$

Thus $\lim_{t\to\infty} f'(t)$ does not exist.

(D) Golden Extreme Value Theorem.

Ans: (D).

Question 15. [03.3-M-★★] Consider the function

$$f(x) = \begin{cases} x\cos x, & \text{if } x \text{ is rational,} \\ \sin x, & \text{if } x \text{ is irrational.} \end{cases}$$

Which of the following statements are **true**?

(A) $f(x)$ is continuous at $x = 0$. (B) $f(x)$ is differentiable at $x = 0$.

(C) $f(x)$ is continuous at infinite many points. (D) $f(x)$ is differentiable at infinite many points.

Solution 15. (A) Notice that $|\sin x| \le |x|$ for all $x \in \mathbb{R}$. Hence $|f(x)| \le |x|$ for all $x \in \mathbb{R}$. $\lim_{x\to 0} |x| = |0| = 0$. Thus $\lim_{x\to 0} f(x) = 0 = f(0)$ by Squeeze Theorem.

(B)

$$\left|\frac{f(x)-f(0)}{x-0} - 1\right| = \begin{cases} |\cos x - 1|, & x \in \mathbb{Q}, \\ \left|\dfrac{\sin x}{x} - 1\right|, & x \notin \mathbb{Q}. \end{cases}$$

So $\lim_{x\to 0} \left|\frac{f(x)-f(0)}{x-0} - 1\right| = 0$, $f'(0) = 1$.

(C) Suppose that f is continuous at $\alpha \in \mathbb{R}$. For any $\{q_n\}_{n=1}^{\infty} \subseteq \mathbb{R}\backslash\mathbb{Q}$ and $\{r_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = \alpha$, we see that $\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} f(r_n) = f(\alpha)$. Thus

$$\begin{cases} \lim_{n\to\infty} r_n \cos r_n & = \alpha\cos\alpha = f(\alpha), \\ \lim_{n\to\infty} \sin q_n & = \sin\alpha = f(\alpha). \end{cases}$$

Therefore, $\alpha\cos\alpha = \sin\alpha$. Let $g(x) = x\cos x - \sin x$. Then for any $k \in \mathbb{Z}$, $g\left(2k\pi - \frac{\pi}{2}\right) = 1 > 0 > g\left(2k\pi + \frac{\pi}{2}\right) = -1$. Clearly, g is continuous on $\mathbb{R}$. By Intermediate Value Theorem, there are $c_k \in \left(2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}\right)$ such that $g(c_k) = 0$. For any $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ such that $\lim_{n\to\infty} x_n = c_k$, $\lim_{n\to\infty} g(x_n) = g(c_k) = 0$. Thus

$$\begin{cases} \lim_{n\to\infty} x_n\cos x_n & = c_k\cos c_k, \\ \lim_{n\to\infty} \sin x_n & = \sin c_k, \\ c_k\cos c_k = \sin c_k & = f(c_k). \end{cases}$$

Hence, $\lim_{n\to\infty} f(x_n) = f(c_k)$, f is continuous at c_k for every $k \in \mathbb{Z}$.

(D) Let f be differentiable at α. Then f is continuous α. Let $\{q_n\}_{n=1}^{\infty} \subseteq \mathbb{R}\backslash\mathbb{Q}$ and $\{r_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = \alpha$. we see that

$$f(\alpha) = \begin{cases} \lim_{n\to\infty} f(q_n) & = \lim_{n\to\infty} \sin q_n = \sin\alpha; \\ \lim_{n\to\infty} f(r_n) & = \lim_{n\to\infty} r_n\cos r_n = \alpha\cos\alpha. \end{cases}$$

Now,

$$f'(\alpha) = \begin{cases} \lim_{n\to\infty} \dfrac{f(q_n)-f(\alpha)}{q_n-\alpha} & = \lim_{n\to\infty} \dfrac{\sin q_n - \sin\alpha}{q_n-\alpha} = (\sin x)'\Big|_{x=\alpha} = \cos\alpha, \\ \lim_{n\to\infty} \dfrac{f(r_n)-f(\alpha)}{r_n-\alpha} & = \lim_{n\to\infty} \dfrac{r_n\cos r_n - \alpha\cos\alpha}{r_n-\alpha} = (x\cos x)'\Big|_{x=\alpha} = \cos\alpha - \alpha\sin\alpha. \end{cases}$$

So $\begin{cases} \sin\alpha &= \alpha\cos\alpha, \\ \cos\alpha &= \cos\alpha - \alpha\sin\alpha. \end{cases}$ $\alpha = 0.$

`Ans:` `(A)(B)(C).`

Question 16. [08.1-S-★] The **length** of the parametric curve $x = \cos\theta$, $y = \theta + \sin\theta$, $\theta \in [0, \pi]$ is

`(A)` 2; `(B)` 4; `(C)` $\sqrt{2}$; `(D)` $2\sqrt{2}$.

Solution 16. The length =

$$\int_0^\pi \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\ d\theta = \int_0^\pi \sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2}\ d\theta = \int_0^\pi \sqrt{2 + 2\cos\theta}\ d\theta$$

$$= \int_0^\pi \sqrt{2 + 2\left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right)}\ d\theta = \int_0^\pi 2\cos\frac{\theta}{2}\ d\theta = 4\sin\frac{\theta}{2}\Big|_{\theta=0}^{\pi} = 4.$$

`Ans:` `(B).`

Question 17. [03.5-S-★] The equation of the **tangent line** to the curve $y\sin(2x) = x\cos(2y)$ at the point $(\pi/2, \pi/4)$ is $y = mx + b$. Then $(m, b) =$

`(A)` $\left(\frac{1}{2}, 0\right)$; `(B)` $\left(0, \frac{\pi}{4}\right)$; `(C)` $\left(-1, \frac{3\pi}{4}\right)$; `(D)` $\left(\frac{1}{\pi}, \frac{\pi}{4} - \frac{1}{2}\right)$.

Solution 17. By Implicit Differentiation,

$$y'\sin(2x) + y \cdot 2\cos(2x) = \cos(2y) + x \cdot y' \cdot (-2)\sin(2y),$$

$$y' = \frac{\cos(2y) - 2y\cos(2x)}{\sin(2x) + 2x\sin(2y)} \quad \text{if} \quad \sin(2x) + 2x\sin(2y) \neq 0.$$

So

$$m = y'\left(\frac{\pi}{2}\right) = \frac{\cos(2y) - 2y\cos(2x)}{\sin(2x) + 2x\sin(2y)}\Big|_{(x,y)=(\pi/2,\pi/4)} = \frac{1}{2},$$

$$b = y - mx|_{(x,y)=(\pi/2,\pi/4)} = 0.$$

`Ans:` `(A).`

Question 18. [03.2-S-★★★★★] Let $f(x)$ and $g(x)$ be polynomials of the **third** degree, and $f(x) - g(x) = x^3 + ax^2 + bx + c$, where a, b, c are real numbers. Assume that $f(x)$ and $g(x)$ are **tangent** to each other at $x = 1$. Moreover, $f(x)$ and $g(x)$ **only intersect** at $x = 1$. Then $(a, b, c) =$

`(A)` $(-3, 3, -1)$; `(B)` $(3, 3, 1)$; `(C)` $(1, -1, -1)$; `(D)` $-1, -1, 1$.

Solution 18. Since $f(x) = g(x)$ if and only if $x = 1$, $f(x) - g(x) = (x-1)p(x)$ for some quadratic polynomial p. Moreover, $f(x)$ and $g(x)$ are tangent to each other at $x = 1$, so $f'(1) = g'(1)$, $f'(1) - g'(1) = p(x) + (x-1)p'(x)|_{x=1} = p(1) = 0$. Thus $p(x) = (x-1)(x-c)$. But $f(x) - g(x) = (x-1)^2(x-c) = 0$ has only one root $x = 1$. Hence $c = 1$, $f(x) - g(x) = (x-1)^3 =^3 -3x^2 + 3x - 1$.

`Ans:` `(A).`

Question 19. [02.3-S-★] The limit $\lim_{x\to 0} \frac{|6x-1| - |6x+1|}{x} =$

`(A)` Dose not exist; `(B)` 12; `(C)` -12; `(D)` 6.

Solution 19. Notice that $6x - 1 > 0$ if and only if $x > \frac{1}{6}$ and $6x + 1 > 0$ if and only if $x > -\frac{1}{6}$. Thus

$$\lim_{x\to 0} \frac{|6x-1| - |6x+1|}{x} = \lim_{x\to 0} \frac{(1-6x) - (6x+1)}{x} = \lim_{x\to 0} \frac{-12x}{x} = -12.$$

`Ans:` `(C).`

Question 20. [06.1-S-★] The line $y = mx$ cuts the region bounded above by the curve $y = x(1-x)$ and below by the x-axis into two parts. Then, the areas of the two parts are **equal** when m is

(A) $\frac{1}{2}$; (B) $1 - \frac{1}{\sqrt[3]{2}}$; (C) $\frac{1}{2} + \frac{1}{2\sqrt{2}}$; (D) $\frac{1}{2} - \frac{1}{2\sqrt{2}}$.

Solution 20. Suppose that $y = mx$ and $y = x(1-x)$ intersects at (α, β) $m\alpha = \alpha(1-\alpha) = \beta$, $\alpha = 0$ or $1-m$. Thus the area of one part is

$$\begin{aligned} &\int_0^{1-m} [x(1-x) - mx]\, dx = \frac{1}{2}\int_0^1 x(1-x)\, dx \\ \Rightarrow &\left(-\frac{1}{3}x^3 + \frac{1-m}{2}x^2 \Big|_{x=0}^{1-m} \right) = \frac{1}{2}\left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_{x=0}^{1} \right) \\ \Rightarrow &\frac{1}{6}(1-m)^3 = \frac{1}{2}\cdot\frac{1}{6} \;\Rightarrow\; m = 1 - \frac{1}{\sqrt[3]{2}}. \end{aligned}$$

Ans: (B).

Future Gadget Laboratory, Department of Applied Mathematics, Science Building A, Guangfu Campus, National Chiao Tung University, Hsinchu city, 30010, Taiwan