SUGGESTED SOLUTION FOR COMPREHENSIVE ASSESSMENT OF CALCULUS (II) IN SUMMER, 2019 (107-2)

TÂN UÍ-TSIȦT

ABSTRACT. This text is a document of suggested solution for comprehensive assessment of Calculus (II), which is held on June 19th, 2019. It happened during the second semester on 107th academical year.

Question 1. [15.3-S-★] The integral of $\int_0^2 \int_{x/2}^1 \sin(y^2)\ dydx$ equals

(A) $\sin 1$; (B) $\cos 1$; (C) $1 - \sin 1$; (D) $1 - \cos 1$.

Solution 1. Set I be the integral. Notice that

$$\begin{cases} 0 \leq x \leq 2, \\ \frac{x}{2} \leq y \leq 1 \end{cases} \Leftrightarrow \begin{cases} 0 \leq x \leq 2, \\ x \leq 2y \leq 2 \end{cases} \Leftrightarrow \begin{cases} 0 \leq y \leq 1, \\ 0 \leq x \leq 2y. \end{cases}$$

By Fubini's theorem,

$$I = \int_0^1 \int_0^{2y} \sin(y^2)\ dxdy = \int_0^1 \left(x \cdot \sin(y^2)\Big|_{x=0}^{2y} \right) dy = \int_0^1 2y \sin(y^2)\ dy.$$

For change of variable, set $u = y^2$. Then $du = 2y\ dy$, and thus $I = \int_{0^2}^{1^2} \sin u\ du = -\cos u\Big|_0^1 = 1 - \cos 1$.

Ans: (D).

Question 2. [11.10-S-★] The first three nonzero terms of the Maclaurin series of $f(x) = \ln \frac{1+x}{1-x}$ are

(A) $2x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5$; (B) $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5$; (C) $2x - \frac{2}{3!}x^3 + \frac{2}{5!}x^5$; (D) $2x - \frac{2}{3}x^3 + \frac{2}{5}x^5$.

Solution 2. For $|x| < 1$,

$$\begin{aligned} f(x) = \ln(1+x) - \ln(1-x) &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \right) - \left(\sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{n+1} x^{n+1} = \sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1} = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \sum_{k=3}^{\infty} \frac{2}{2k+1} x^{2k+1} \end{aligned}$$

Ans: (B).

Question 3. [13.2-S-★] Which of the following is the **tangent line** to the parametric curve $\gamma(t) = \langle te^t, t - 2\ln t \rangle$ at $\gamma(1)$?

(A) $ex + y = e^2 + e$; (B) $x + y = e + 1$; (C) $x + ey = 2e$; (D) $x + 2ey = 3e$.

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Solution 3. First, $\gamma(1) = \langle e, 1\rangle$, and $\gamma'(1) = \left\langle (t+1)e^t, 1 - \frac{2}{t} \right\rangle\Big|_{t=1} = \langle 2e, -1\rangle$. So the tangent line is

$$\langle x, y\rangle - \gamma(1) = s\gamma'(1),\ s \in \mathbb{R} \Leftrightarrow \begin{cases} x - e = 2es, \\ y - 1 = -s, \end{cases} \quad s \in \mathbb{R} \Leftrightarrow (x - e) + 2e(y - 1) = 0 \Leftrightarrow x + 2ey = 3e.$$

`Ans: (D).`

Question 4. [14.7-S-★★★] Find the **smallest** value of α such that the inequality $(x+2y+3z)^4 \leq \alpha(x^4+2y^4+3z^4)$ holds for all real numbers x, y, z.
(A) 6^3; (B) 5^3; (C) 4^3; (D) 3^3.

Solution 4. Let $\alpha \in \mathbb{R}$ and $f(x, y, z) = \alpha(x^4 + 2y^4 + 3z^4) - (x + 2y + 3z)^4$ for all $(x, y, z) \in \mathbb{R}^3$. Now we need to find $\alpha \in \mathbb{R}$ such that $f(x, y, z) \geq 0$. If $\min\limits_{\mathbb{R}^3} f(x, y, z) := m$ exists, then $m \geq 0$.

To find m, we may solve $\nabla f(x, y, z) = \langle 0, 0, 0\rangle$. Thus

$$\begin{aligned} &\nabla f(x, y, z) = 4\left\langle \alpha x^3 - (x + 2y + 3z)^3, 2\left[\alpha y^3 - (x + 2y + 3z)^3\right], 3\left[\alpha z^3 - (x + 2y + 3z)^3\right]\right\rangle = \langle 0, 0, 0\rangle \\ &\Leftrightarrow \alpha x^3 = \alpha y^3 = \alpha z^3 = (x + 2y + 3z)^3. \end{aligned}$$

<u>`Case I:` $\alpha = 0 = (x + 2y + 3z)^3$.</u> Thus $f(x, y, z) = -(x + 2y + 3z)^4$. But $f(1, 0, 0) = -1 < 0$. Contradiction.

<u>`Case II:` $\alpha \neq 0$.</u> So $x^3 = y^3 = z^3$, $x = y = z$. $\alpha x^3 = (x + 2x + 3x)^3 = 6^3x^3$, $\alpha = 6^3$. By Cauchy-Schwarz inequality,

$$\begin{aligned} f(x, y, z) &= 6^2\left[(1 + 2 + 3)(x^4 + 2y^4 + 3z^4)\right] - (x + 2y + 3z)^4 \\ &\geq 6^2\left(x^2 + 2y^2 + 3z^2\right)^2 - (x + 2y + 3z)^4 = \left[(1 + 2 + 3)(x^2 + 2y^2 + 3z^2)\right]^2 - (x + 2y + 3z)^4 \\ &\geq \left[(x + 2y + 3z)^2\right]^2 - (x + 2y + 3z)^4 = 0. \end{aligned}$$

`Ans: (A).`

Question 5. [10.1-S-★] The **maximum** value of $x^2 + y^2$ over the circle $(x + 1)^2 + (y + 1)^2 = 4$ equals
(A) $4 + 4\sqrt{2}$; (B) 10; (C) $5 + 4\sqrt{2}$; (D) $6 + 4\sqrt{2}$.

Solution 5. [Way I] Let $(x, y) = (-1 + 2\cos\theta, -1 + 2\sin\theta)$, $\theta \in \mathbb{R}$.

$$x^2 + y^2 = (-1 + 2\cos\theta)^2 + (-1 + 2\sin\theta)^2 = 6 - 4(\cos\theta + \sin\theta) = 6 - 4\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right).$$

Thus $\max\limits_{\theta\in\mathbb{R}}(x^2 + y^2) = 6 - 4\sqrt{2}\sin\left(-\frac{3\pi}{4} + \frac{\pi}{4}\right) = 6 + 4\sqrt{2}$.

[Way II] We apply the method of Lagrange multiplier to find maximum at point P. Let $f(x, y) = x^2 + y^2$ and $g(x, y) = (x+1)^2 + (y+1)^2$. Note that $\nabla f(x, y) = (2x, 2y)$ and $\nabla g(x, y) = (2(x+1), 2(y+1))$. As $\nabla g(x, y) \neq (0, 0)$ for $g(x, y) = 4$, P satisfies

$$\begin{cases} \nabla f(x, y) = & \lambda\nabla g(x, y), \\ \quad g(x, y) = & 4. \end{cases}$$

Since $(x, y) \neq (0, 0)$, we know that $\nabla f(x, y) \neq (0, 0)$, and λ can not be zero. So $x = y = \frac{\lambda}{1 - \lambda}$ with $\lambda \neq 1$, it gives $(x, y) = (-1, -1) \pm (\sqrt{2}, \sqrt{2})$. Furthermore, $\max\limits_{g(x,y)=1} f(x, y) = f(-1 - \sqrt{2}, -1 - \sqrt{2}) = 6 + 4\sqrt{2}$.

`Ans: (D).`

Question 6. [15.6-S-★★★] Let S be the portion of the hemisphere $f(x, y) = \sqrt{25 - x^2 - y^2}$ that lies above the disk $x^2 + y^2 \leq 9$. Then, the **surface area** of S equals
(A) π; (B) 10π; (C) 5π; (D) 12π.

Solution 6. The surface area =

$$\iint_{x^2+y^2\le 9} \sqrt{1+(f_x)^2+(f_y)^2}\, dA = \iint_{x^2+y^2\le 9} \sqrt{1+\left(\frac{1}{2}\frac{(-2x)}{\sqrt{25-x^2-y^2}}\right)^2+\left(\frac{1}{2}\frac{(-2y)}{\sqrt{25-x^2-y^2}}\right)^2}\, dA$$
$$= \iint_{x^2+y^2\le 9} \sqrt{1+\frac{x^2+y^2}{25-x^2-y^2}}\, dA = \iint_{x^2+y^2\le 9} \frac{5}{\sqrt{25-x^2-y^2}}\, dA.$$

Now change of variable for polar coordinate, set $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$S = \int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25-r^2}} r\, drd\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 \frac{5}{\sqrt{25-r^2}} r\, dr\right)$$
$$= 2\pi\cdot\left(-5\sqrt{25-r^2}\Big|_{r=0}^3\right) = 2\pi\cdot(-5)\cdot\left(\sqrt{25-3^2}-\sqrt{25-0^2}\right) = 2\pi\cdot(-5)\cdot(4-5) = 10\pi.$$

`Ans: (B).`

Question 7. [14.3-S-★★★★★] Suppose $f(x,y)$ is a continuous function satisfying

$$\lim_{(x,y)\to(1,2)} \frac{f(x,y)-x^2-y^2}{\sqrt{(x-1)^2+(y-2)^2}} = 0.$$

Then, $f_x(1,2)$ equals

(A) 5; (B) 4; (C) 2; (D) ∞.

Solution 7.

$$\lim_{x\to 1}\frac{f(x,2)-x^2-2^2}{\sqrt{(x-1)^2+(2-2)^2}} = \lim_{x\to 1}\frac{f(x,2)-x^2-4}{|x-1|} = 0$$
$$\Rightarrow \lim_{x\to 1}\left|\frac{f(x,2)-x^2-4}{|x-1|}\right| = \lim_{x\to 1}\left|\frac{f(x,2)-x^2-4}{x-1}\right| = |0| = 0 \;\Rightarrow\; \lim_{x\to 1}\frac{f(x,2)-x^2-4}{x-1} = 0.$$

Notice that $\lim_{x\to 1} x-1 = 0$, $\lim_{x\to 1} f(x,2)-x^2-4 = 0$, $\lim_{x\to 1} f(x,2) = \lim_{x\to 1} x^2+4 = 5$. Since f is continuous at $(1,2)$, $\lim_{x\to 1} f(x,2) = f(1,2) = 5$. Finally,

$$f_x(1,2) = \lim_{x\to 1}\frac{f(x,2)-f(1,2)}{x-1} = \lim_{x\to 1}\frac{f(x,2)-5}{x-1}$$
$$= \lim_{x\to 1}\frac{(f(x,2)-x^2-4)+(x^2+4-5)}{x-1} = 0 + \lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2.$$

`Ans: (C).`

Question 8. [15.4-S-★★] Let E be the solid bounded by the sphere $x^2+y^2+z^2=4$ and the cylinder $x^2+(y-1)^2=1$. Then, the **volume** of E is

(A) $\frac{16}{9}(3\pi-1)$; (B) $\frac{16}{9}(3\pi-2)$; (C) $\frac{16}{9}(3\pi-4)$; (D) $\frac{16}{9}(3\pi-8)$.

Solution 8. By the polar coordinate on $\mathbb{R}^2$. Notice that

$$D = \{(x,y)\in\mathbb{R}^2 \mid x^2+(y-1)^2\le 1\} = \{(r,\theta) \mid 0\le\theta\le\pi,\ 0\le r\le 2\sin\theta\},$$

and $x^2+y^2+z^2 \le 4 \Leftrightarrow -\sqrt{4-(x^2+y^2)} \le z \le \sqrt{4-(x^2+y^2)}$. So the volume of E is equal to

$$\iint_{(x,y)\in E} \int_{-\sqrt{4-(x^2+y^2)}}^{\sqrt{4-(x^2+y^2)}} dzdA = \int_0^\pi \int_0^{2\sin\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ dzdrd\theta = \int_0^\pi \int_0^{2\sin\theta} 2r\sqrt{4-r^2}\ drd\theta$$

$$= \int_0^\pi \left(-\frac{2}{3}\left(4-r^2\right)^{3/2}\Big|_{r=0}^{2\sin\theta} \right) d\theta = -\frac{2}{3}\int_0^\pi \left(4-4\sin^2\theta\right)^{3/2} - 8\ d\theta = -\frac{2}{3}\int_0^\pi 8|\cos\theta|^3 - 8\ d\theta$$

$$= \left(-\frac{2}{3}\right)\cdot 2\cdot 8\int_0^{\pi/2} \cos^3\theta - 1\ d\theta = -\frac{32}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta - \theta\Big|_{\theta=0}^{\pi/2}\right) = -\frac{32}{3}\left(1-\frac{1}{3}-\frac{\pi}{2}\right) = \frac{16}{9}(3\pi-4).$$

`Ans: (C).`

Question 9. [15.10-S-★★] Let $T(u,v) = (2019u+2020v, 2017u+2018v)$ and R be the trapezoidal with vertices $(0,0)$, $(4,0)$, $(1,2)$ and $(3,2)$. Suppose S is a region such that $T(S)=R$. Then, the **area** of S equals

(A) 3; (B) 6; (C) 12; (D) 24.

Solution 9. Notice that area of $R = (\|(3,2)-(1,2)\| + \|(4,0)-(0,0)\|)\cdot 2 \div 2 = 6$. Let $x = 2019u+2020v$ and $y = 2017u+2018v$. On the other hand, area of $R =$

$$\iint_R dxdy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_S \begin{vmatrix} 2019 & 2020 \\ 2017 & 2018 \end{vmatrix} dudv = \iint_S 2\ dudv = 2\cdot(\text{area of } S).$$

Therefore, area of $S = \frac{6}{2} = 3$.

`Ans: (A).`

Question 10. [10.4-S-★★] Consider the polar curve $r = 1+2\cos\theta$ and let R be the region inside the larger loop but outside the smaller loop. Then, the **area** is

(A) $\pi+\sqrt{3}$; (B) $\pi+2\sqrt{3}$; (C) $\pi+3\sqrt{3}$; (D) $\pi+4\sqrt{3}$.

Solution 10. Notice that $r(\theta+2\pi) = r(\theta) = r(-\theta)$. Thus we can consider the question on $-\pi\le\theta\le\pi$ only, and we can discover that the graph is symmetric with respect to the x-axis.

For $-\pi\le\theta\le\pi$, $y(\theta) = r\sin\theta = (1+2\cos\theta)\sin\theta \ge 0$ if and only if $(-\pi\le\theta\le -\frac{2\pi}{3}$ or $0\le\theta\le\frac{2\pi}{3})$. Here we deduce that the upper-larger loop between $\theta=0$ and $\theta=\frac{2\pi}{3}$, and the the upper-smaller loop between $\theta=-\pi$ and $\theta=-\frac{2\pi}{3}$.

Thus the desired area =

$$2\left(\int_0^{2\pi/3} \frac{1}{2}r^2\ d\theta - \int_{-\pi}^{-2\pi/3}\frac{1}{2}r^2\ d\theta\right) = \int_0^{2\pi/3}(1+2\cos\theta)^2\ d\theta - \int_{-\pi}^{-2\pi/3}(1+2\cos\theta)^2\ d\theta$$

$$= \int_0^{2\pi/3} 1+4\cos\theta+4\cos^2\theta\ d\theta - \int_{-\pi}^{-2\pi/3} 1+4\cos\theta+4\cos^2\theta\ d\theta$$

$$= \int_0^{2\pi/3} 4\cos\theta+2\cos(2\theta)+3\ d\theta - \int_{-\pi}^{-2\pi/3} 4\cos\theta+2\cos(2\theta)+3\ d\theta$$

$$= \left(4\sin\theta+\sin(2\theta)+3\theta|_0^{2\pi/3}\right) - \left(4\sin\theta+\sin(2\theta)+3\theta|_{-\pi}^{-2\pi/3}\right)$$

$$= \left(\frac{3\sqrt{3}}{2}+2\pi\right) - \left[\left(\frac{-3\sqrt{3}}{2}-2\pi\right)-(-3\pi)\right] = 3\sqrt{3}+\pi.$$

`Ans: (C).`

Question 11. [11.9-M-★★] Let $f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(2n)!}$. Which of the following falls in the **range** of f?

(A) $\frac{-1}{2}$; (B) $\frac{1}{2}$; (C) $\frac{3}{2}$; (D) $\frac{5}{2}$.

Solution 11. [Way I]

$$f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!}\bigg|_{y=x^2} = \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{y^k}{k!} + \sum_{k=0}^{\infty}\frac{(-y)^k}{k!}\right)\bigg|_{y=x^2} = \frac{1}{2}\left(e^y + e^{(-y)}\right)\bigg|_{y=x^2} = \frac{1}{2}\left(e^{x^2} + e^{-x^2}\right)$$

for all $x \in \mathbb{R}$.

By inequality of arithmetic and geometric means, $f(x) \geq \sqrt{e^{x^2} \cdot e^{-x^2}} = \sqrt{1}$. Now, $f(0) = 1$ and $\lim_{x\to\infty} f(x) = \infty$, by intermediate value theorem, the range of $f = f([0,\infty)) = [1,\infty)$.

[Way II] Fixed $x \in \mathbb{R}\backslash\{0\}$. Then

$$\lim_{n\to\infty}\left|\frac{\frac{x^{4(n+1)}}{(2(n+1))!}}{\frac{x^{4n}}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{x^4}{(2n+1)(2n+2)}\right| = 0.$$

Thus $\sum_{n=0}^{\infty}\frac{x^{4n}}{(2n)!}$ converges. By theorem of continuity of power series, f is continuous on $\mathbb{R}$.

Now, $f(x) \geq 1 + \frac{x^4}{2!} \geq 1 = f(0)$ and $\lim_{x\to\pm\infty} 1 + \frac{x^4}{2!} = \infty$. By intermediate value theorem, the range of $f = f([0,\infty)) = [1,\infty)$.

`Ans: (C)(D).`

Question 12. [11.9-M-★★] Assume that $\sum_{n=1}^{\infty} a_n x^n$ is convergent at $x = 1$ and divergent at $x = 2$. Which of the following statements are **FALSE**?

(A) $\lim_{n\to\infty} a_n = 0$. (B) $\sum_{n=1}^{\infty} a_n 2^{-n}$ is absolutely convergent.

(C) $\sum_{n=1}^{\infty} a_n 3^n$ is convergent. (D) $\sum_{n=1}^{\infty} a_n n x^{n-1}$ is convergent at $x = 3$.

Solution 12. Let I be the interval of convergence of $\sum_{n=1}^{\infty} a_n x^n$, then

$$(\ 0 - |1-0|, 0 + |1-0|\ ] = (-1, 1] \subseteq I \subseteq [\ 0 - |2-0|, 0 + |2-0|\ ) = [-2, 2).$$

(A) $\sum_{n=1}^{\infty} a_n \cdot 1^n = \sum_{n=1}^{\infty} a_n$ converges absolutely. By divergence test, $\lim_{n\to\infty} a_n = 0$.

(B) $\frac{1}{2} \in I$ and $\frac{1}{2}$ is not an endpoint of I (the endpoint of I are $\pm c$, $1 \leq c \leq 2$). Thus this series converges.

(C) $3 \notin I$.

(D) $\sum_{n=1}^{\infty} a_n \cdot n x^{n-1} = \sum_{n=1}^{\infty}\left(\frac{d}{dx} a_n x^n\right)$ converges on J, where $J \subseteq I \subseteq [-2, 2)$. Thus this power series diverges at $x = 3$.

`Ans: (C)(D).`

Question 13. [14.6-M-★★★★★] Assume that $f(x, y)$ is differentiable and set $g(t) = f(at, bt)$, where a and b are constants. Which of the following are **TRUE**?

(A) If, for any $a, b \in \mathbb{R}$, g has a local maximum at 0, then f has a local maximum at $(0, 0)$.

(B) If, for any $a, b \in \mathbb{R}$, g has a local minimum at 0, then $\nabla f(0, 0) = \langle 0, 0\rangle$.

(C) If $\nabla f(0, 0) = \langle 0, 0\rangle$, then g has a local minimum at 0 for any $a, b \in \mathbb{R}$.

(D) If f has a local maximum at $(0, 0)$, then g has a local maximum at 0 for any $a, b \in \mathbb{R}$.

Solution 13. (A) Let $f(x,y)=\cos(y-x^2)+y^4$. Fix $a,b\in\mathbb{R}$. Then $g(t)=\cos(bt-a^2t^2)+b^4t^4$. Hence

$$\begin{aligned} g'(0) &= -(b-2a^2t)\sin(bt-a^2t^2)+4b^4t^3\big|_{t=0}=0,\\ g''(0) &= 2a^2\sin(bt-a^2t^2)-(b-2a^2t)^2\cos(bt-a^2t^2)+12b^4t^2\big|_{t=0}=-b^2. \end{aligned}$$

Thus g has local maxima at 0 while $b\neq 0$. While $b=0$, $g(t)=\cos(-a^2t^2)\le 1=g(0)$, g has local maxima at 0. In summary, g has local maxima at 0 for any $a,b\in\mathbb{R}$.

But $f(t,t^2)=t^8>0=f(0,0)$ for any $t>0$, f has no local maxima at 0.

(B)

$$g'(0)=0=\frac{d}{dt}f(at,bt)\bigg|_{t=0}=\nabla f(at,bt)\cdot\langle a,b\rangle|_{t=0}=\nabla f(0,0)\cdot\langle a,b\rangle\quad\forall\ \langle a,b\rangle\in\mathbb{R}^2\ \Rightarrow\ \nabla f(0,0)=\langle 0,0\rangle.$$

(C) Let $f(x,y)=x^3$. Then $\nabla f(0,0)=\langle 3x^2,0\rangle\big|_{t=0}\langle 0,0\rangle$. But $f(t,0)=t^3>f(0,0)=0>f(-t,0)=-t^3$ for all $t>0$. f has no local extreme value at 0.

(D) There is a $\delta>0$ such that $f(0,0)\ge f(x,y)$ for all $(x,y)\in\mathbb{R}^2$ with $\|(x,y)-(0,0)\|=\sqrt{x^2+y^2}<\delta$. Thus for $\tilde{\delta}:=\dfrac{\delta}{\sqrt{a^2+b^2}+1}$ and $|t|<\tilde{\delta}$, we have $\|(at,bt)-(0,0)\|=|t|\sqrt{a^2+b^2}<\delta$, $f(0,0)\ge f(at,bt)=g(t)$.

Ans: (B)(D).

Question 14. [14.6-M-★★★] Let $f(x,y,z)=e^x-y^4-z^3+xyz$ and $g(t)=f(at,1+bt,2+ct)$. Suppose there is $\varepsilon>0$ such that g is **decreasing** on $(0,\varepsilon)$. Then, (a,b,c) can be

(A) $(-3,4,12)$; (B) $(3,-4,-12)$; (C) $(0,3,-1)$; (D) $(0,-3,1)$.

Solution 14. Clearly, $f\in\mathcal{C}^2(\mathbb{R}^3)$. Thus $g\in\mathcal{C}^2(\mathbb{R})$, $g'(t)=\nabla f(at,1+bt,2+ct)\cdot\langle a,b,c\rangle$. So

$$g'(0)=\nabla f(0,1,2)\cdot\langle a,b,c\rangle=\langle e^x+yz,-4y^3+xz,-3z^2+xy\rangle\big|_{(x,y,z)=(0,1,2)}\cdot\langle a,b,c\rangle=\langle 3,-4,-12\rangle\cdot\langle a,b,c\rangle.$$

Moreover,

$$\begin{aligned} g''(t) &= \frac{d}{dt}\left(af_x(at,1+bt,2+ct)+bf_y(at,1+bt,2+ct)+cf_z(at,1+bt,2+ct)\right)\\ &= a\nabla f_x(at,1+bt,2+ct)\cdot\langle a,b,c\rangle+b\nabla f_y(at,1+bt,2+ct)\cdot\langle a,b,c\rangle+c\nabla f_z(at,1+bt,2+ct)\cdot\langle a,b,c\rangle\\ &= a^2e^x-12b^2y^2-6c^2z+2abz+2acy+2bcx\big|_{x=at,\ b=1+bt,\ z=2+ct}. \end{aligned}$$

(A) $g'(0)=-3^2-4^2-12^2<0$. Since $g\in\mathcal{C}^1(\mathbb{R})$, there is a $\delta>0$ such that $g'(t)<0$ for all $t\in[0,\delta)$. Hence g is strictly decreasing on $(0,\delta)$.

(B) $g'(0)=3^2+(-4)^2+(-12)^2>0$. Since $g\in\mathcal{C}^1(\mathbb{R})$, there is a $\delta>0$ such that $g'(t)>0$ for all $t\in[0,\delta)$. Hence g is strictly increasing on $(0,\delta)$.

(C) $g'(0)=\langle 3,-4,-12\rangle\cdot\langle 0,3,-1\rangle=0$. Thus 0 is a critical point. Now,

$$g''(0)=a^2-12b^2-12c^2+4ab+2ac\big|_{(a,b,c)=(0,3,-1)}=-120<0.$$

So g has a local maxima at 0.

(D) $g'(0)=\langle 3,-4,-12\rangle\cdot\langle 0,-3,1\rangle=0$. Thus 0 is a critical point. Now,

$$g''(0)=a^2-12b^2-12c^2+4ab+2ac\big|_{(a,b,c)=(0,-3,1)}=-120<0.$$

So g has a local maxima at 0.

Ans: (A)(C)(D).

Question 15. [15.10-M-★★] Let f be the following function.

$$f(t,s)=\int_{-t}^{t}\int_{-s\sqrt{1-(x/t)^2}}^{s\sqrt{1-(x/t)^2}}(x^2+y^2)\ dydx.$$

Which of the followings are **TRUE**?

(A) $f(1,1)=\dfrac{\pi}{2}$; (B) $f_t(2,1)=3\pi$; (C) $f_s(3,1)=9\pi$; (D) $f_{ts}(4,1)=12\pi$.

Solution 15. Notice that

$$-s\sqrt{1-\left(\frac{x}{t}\right)^2}\le y\le s\sqrt{1-\left(\frac{x}{t}\right)^2}\Leftrightarrow\left|\frac{y}{s}\right|\le\sqrt{1-\left(\frac{x}{t}\right)^2}\Leftrightarrow\left(\frac{x}{t}\right)^2+\left(\frac{y}{s}\right)^2\le 1$$

if $s,t>0$. Thus

$$f(t,s)=\iint_D (x^2+y^2)\ dydx.$$

where $D=\left\{(x,y)\in\mathbb{R}^2\ \middle|\ \left(\frac{x}{t}\right)^2+\left(\frac{y}{s}\right)^2\le 1\right\}$. Let $\begin{cases}x=tr\cos\theta,\\ Y=sr\sin\theta.\end{cases}$ Then

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=\begin{vmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}t\cos\theta&-tr\sin\theta\\ s\sin\theta&sr\cos\theta\end{vmatrix}=tsr.$$

By change of variable,

$$\begin{aligned}f(t,s)&=\int_0^{2\pi}\int_0^1(t^2r^2\cos^2\theta+s^2r^2\sin^2\theta)tsr\ drd\theta=\frac{ts}{4}\int_0^{2\pi}(t^2\cos^2\theta+s^2\sin^2\theta)\ d\theta\\ &=\frac{ts}{4}\int_0^{2\pi}\left[t^2\left(\frac{\cos 2\theta+1}{2}\right)+s^2\left(\frac{1-\cos 2\theta}{2}\right)\right]\ d\theta\\ &=\frac{ts}{4}\left[\frac{t^2}{4}\sin 2\theta-\frac{s^2}{4}\sin 2\theta+\left(\frac{s^2+t^2}{2}\right)\theta\right]\Bigg|_{\theta=0}^{2\pi}=\frac{\pi}{4}ts(t^2+s^2).\end{aligned}$$

Hence $\begin{cases}f_t(t,s)=\frac{\pi}{4}(3t^2s+s^3),\\ f_s(t,s)=\frac{\pi}{4}(t^3+3ts^2),\\ f_{ts}(t,s)=\frac{3\pi}{4}(t^2+s^2).\end{cases}$

Ans: (A)(C).

Question 16. [14.8-M-★] Let $f(x,y)=\dfrac{1}{x}+\dfrac{1}{y}$ for $x\neq 0$ and $y\neq 0$. Assume that, under the constraint $\dfrac{1}{x^2}+\dfrac{1}{y^2}=1$, f attains its **extreme values** at P. Find P.

(A) $\langle\sqrt{2},\sqrt{2}\rangle$; (B) $\langle-\sqrt{2},-\sqrt{2}\rangle$; (C) $\left\langle 2,\dfrac{2}{\sqrt{3}}\right\rangle$; (D) $\left\langle -2,-\dfrac{2}{\sqrt{3}}\right\rangle$.

Solution 16. [Way I] First, we apply the method of Lagrange multiplier to find P. Set $g(x,y)=x^{-2}+y^{-2}$. Note that

$$\nabla f(x,y)=-\left\langle x^{-2},y^{-2}\right\rangle,\qquad \nabla g(x,y)=-2\left\langle x^{-3},y^{-3}\right\rangle.$$

As $\nabla g(x,y)\neq\langle 0,0\rangle$ for $g(x,y)=1$, P satisfies

$$\begin{cases}\nabla f(x,y)=&\lambda\nabla g(x,y),\\ g(x,y)=&1.\end{cases}$$

Since $\nabla f(x,y)\neq(0,0)$, λ can not be zero. So $2\lambda=x=y$, it gives $(x,y)=\pm(\sqrt{2},\sqrt{2})$. Furthermore, $\max\limits_{g(x,y)=1}f(x,y)$ $=f(\sqrt{2},\sqrt{2})=\sqrt{2}$, and $\min\limits_{g(x,y)=1}f(x,y)=f(-\sqrt{2},-\sqrt{2})=-\sqrt{2}$.

[Way II] By Cauchy-Schwarz inequality. For $x,y\neq 0$, $\left(\dfrac{1}{x}+\dfrac{1}{y}\right)^2\le\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)(1^2+1^2)$. Thus

$$-\sqrt{2}\le\frac{1}{x}+\frac{1}{y}\le\sqrt{2}.$$

Furthermore,

$$\frac{1}{x}+\frac{1}{y}=\sqrt{2} \;\Leftrightarrow\; \frac{1}{x}=\frac{1}{y}=t>0 \;\Leftrightarrow\; \frac{1}{x^2}=\frac{1}{y^2}=\frac{1}{2} \text{ and } x,y>0$$

$$\Leftrightarrow\; P=(x,y)=\left(\frac{1}{1/\sqrt{2}},\frac{1}{1/\sqrt{2}}\right)=\left(\sqrt{2},\sqrt{2}\right), \quad \text{and}$$

$$\frac{1}{x}+\frac{1}{y}=-\sqrt{2} \;\Leftrightarrow\; \frac{1}{x}=\frac{1}{y}=t<0 \;\Leftrightarrow\; \frac{1}{x^2}=\frac{1}{y^2}=\frac{1}{2} \text{ and } x,y<0$$

$$\Leftrightarrow\; P=(x,y)=\left(\frac{1}{-1/\sqrt{2}},\frac{1}{-1/\sqrt{2}}\right)=\left(-\sqrt{2},-\sqrt{2}\right).$$

[Way III] Set

$$\begin{cases} \frac{1}{x}=\cos\theta, \\ \frac{1}{y}=\sin\theta, \end{cases} \quad \theta\neq\frac{n\pi}{2},\ n\in\mathbb{Z}.$$

So $\frac{1}{x}+\frac{1}{y}=\cos\theta+\sin\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\leq\sqrt{2}$. The equality holds if and only if $\theta=\frac{\pi}{4}+2n\pi$, $n\in\mathbb{Z}$. Thus $P=(x,y)=\left(\frac{1}{\cos(\pi/4)},\frac{1}{\sin(\pi/4)}\right)=\left(\sqrt{2},\sqrt{2}\right)$.

On the other hand, $\frac{1}{x}+\frac{1}{y}=\cos\theta+\sin\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\geq-\sqrt{2}$. The equality holds if and only if $\theta=-\frac{\pi}{4}+2n\pi$, $n\in\mathbb{Z}$. Thus $P=(x,y)=\left(\frac{1}{\cos(-\pi/4)},\frac{1}{\sin(-\pi/4)}\right)=\left(-\sqrt{2},-\sqrt{2}\right)$.

`Ans: (A)(B).`

Question 17. [14.5-M-★★] Let $f(x,y)=\frac{1}{x}+\frac{1}{y}$ for $x,y\neq0$. Suppose that, under the constraint $\frac{1}{x^2}+\frac{1}{y^2}=1$, f attains its extreme values at P. Assume that, on the parametric curve $\langle t,at^2+bt\rangle$, f has **extreme value** at P. Find the values of a,b.

(A) $\left\langle\frac{\sqrt{2}}{3},\frac{1}{3}\right\rangle$; (B) $\left\langle-\sqrt{2},3\right\rangle$; (C) $\left\langle-\frac{\sqrt{2}}{3},\frac{1}{3}\right\rangle$; (D) $\left\langle\sqrt{2},3\right\rangle$.

Solution 17. Set $\gamma(t)=\langle t,at^2+bt\rangle$ and $h(t)=f(\gamma(t))=t^{-1}+(at^2+bt)^{-1}$. Since P sits on the parametric curve γ, $\pm\left\langle\sqrt{2},\sqrt{2}\right\rangle=\gamma(t_0)$ for some $t_0\in\mathbb{R}$. This implies

$$\begin{cases} 2a+\sqrt{2}b &= \sqrt{2}, \text{ if } t_0=\sqrt{2}; \\ 2a-\sqrt{2}b &= -\sqrt{2}, \text{ if } t_0=-\sqrt{2}. \end{cases}$$

By chain rule for higher dimension,

$$h'(t)=\nabla f(\gamma(t))\cdot\langle1,2at+b\rangle=-\frac{1}{t^2}-\frac{2at+b}{(at^2+bt)^2}.$$

By Fermat's theorem, as P is a local minimum of f over $\gamma(t)$, one has $h'(\sqrt{2})=0$. This implies the following two systems

$$\begin{cases} 2a+\sqrt{2}b &= \sqrt{2}, \\ 2\sqrt{2}a+b &= -1, \end{cases} \quad \text{for } t_0=\sqrt{2}; \qquad \text{and} \qquad \begin{cases} 2a-\sqrt{2}b &= -\sqrt{2}, \\ -2\sqrt{2}a+b &= -1, \end{cases} \quad \text{for } t_0=-\sqrt{2},$$

lead to $\langle t_0,a,b\rangle=\left\langle\sqrt{2},-\sqrt{2},3\right\rangle$ or $\left\langle-\sqrt{2},\sqrt{2},3\right\rangle$. Now, for $t\neq0$, $h''(t)=2\left[\frac{1}{t^3}+\frac{3a^2t^2+3abt+b^2}{(at^2+bt)^3}\right]$. So

$$h''(t_0)|_{\langle t_0,a,b\rangle=\left\langle\sqrt{2},-\sqrt{2},3\right\rangle}=2\sqrt{2}>0, \quad \text{and} \quad h''(t_0)|_{\langle t_0,a,b\rangle=\left\langle-\sqrt{2},\sqrt{2},3\right\rangle}=-2\sqrt{2}<0.$$

In summary, h has local minima at $P=\left(\sqrt{2},\sqrt{2}\right)$ on $\left\{\left(t,-\sqrt{2}t^2+3t\right)\mid t\in\mathbb{R}\backslash\{0,3/\sqrt{2}\}\right\}$, and local maxima at $P=\left(-\sqrt{2},-\sqrt{2}\right)$ on $\left\{\left(t,\sqrt{2}t^2+3t\right)\mid t\in\mathbb{R}\backslash\{0,-3/\sqrt{2}\}\right\}$, respectively.

`Ans: (B)(D).`

Question 18. [11.1-S-★] Determine the **convergency** of the sequence of $\{|a_n|^{1/n}\}_{n=1}^{\infty}$ where

$$a_n = \begin{cases} 2^{-n+\sqrt{n}}, & \text{if } n \text{ is even;} \\ 3^{-n+\sqrt{n}}, & \text{if } n \text{ is odd .} \end{cases}$$

(A) It diverges. (B) It converges to $\frac{1}{2}$. (C) It converges to $\frac{1}{3}$. (D) It converges to 1.

Solution 18.

$$\lim_{n\to\infty} |a_{2n}|^{1/(2n)} = \lim_{n\to\infty} 2^{(-1+1/\sqrt{2n})} = \lim_{n\to\infty} 2^{(-1+0)} = \frac{1}{2},$$
$$\lim_{n\to\infty} |a_{2n+1}|^{1/(2n+1)} = \lim_{n\to\infty} 3^{(-1+1/\sqrt{2n+1})} = \lim_{n\to\infty} 3^{(-1+0)} = \frac{1}{3}.$$

There are two subsequences converge to different limits, thus the limit of $\{|a_n|^{1/n}\}_{n=1}^{\infty}$ does not exist.

`Ans: (A).`

Question 19. [11.4-M-★★] The following claim is **TRUE**:

"If there are some real $0 < q < 1$ and integer $N \geq 0$ such that $|b_n|^{1/n} \leq q$ for all $n \geq N$, then $\sum_{n=1}^{\infty} b_n$ converges."

For the following series, choose the ones that can be determined the convergency by the claim above.

(A) $\sum_{n=1}^{\infty} \frac{n^2}{(1.5)^n}$; (B) $\sum_{n=1}^{\infty} \frac{1}{n!}$; (C) $\sum_{n=1}^{\infty} \frac{1}{n^2}$; (D) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{2^n(n!)}$.

Solution 19. (A) Choose $r = 1.2$; notice that $1 < r < 1.5$. Since $\lim_{x\to\infty} \frac{x^2}{r^x} = 0$ by L'Hopital's rule, there is an $M \in \mathbb{N}$ such that $0 < \frac{x^2}{r^x} < 1$ when $x > M$. Hence for all $n \geq M$,

$$\left|\frac{n^2}{(1.5)^n}\right|^{1/n} < \left(\frac{r^n}{(1.5)^n}\right)^{1/n} = 0.8 < 1.$$

(B) For $n \geq 4$,

$$n! = (n \cdot (n-1) \cdot \ldots \cdot 5) \cdot 4! \geq 2^{n-4} \cdot 2^4 = 2^n.$$

Thus $\frac{1}{n!} \leq \left(\frac{1}{2}\right)^n$ for $n \geq 4$.

(C) Since $\lim_{x\to\infty} \sqrt[x]{x} = 1$ by L'Hopital's rule, $\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = 1$. For every $q < 1$, there is an $M \in \mathbb{N}$ such that $\left(\frac{1}{n^2}\right)^{1/n} > q$ when $n > M$.

(D) Note that for all $n \in \mathbb{N}$,

$$\frac{1\cdot 3\cdot 5\cdot 7\cdot \ldots \cdot (2n-3)\cdot (2n-1)}{2\cdot 4\cdot 6\cdot 8\cdot \ldots \cdot (2n-2)\cdot 2n} \geq \frac{1\cdot 2\cdot 4\cdot 6\cdot \ldots \cdot (2n-4)\cdot (2n-2)}{2\cdot 4\cdot 6\cdot 8\cdot \ldots \cdot (2n-2)\cdot 2n} = \frac{1}{2n}.$$

By p- series test and comparison test, it diverges.

`Ans: (A)(B).`

Question 20. [11.6-M-★★] Let

$$a_n = \begin{cases} 2^{-n+\sqrt{n}}, & \text{if } n \text{ is even;} \\ 3^{-n+\sqrt{n}}, & \text{if } n \text{ is odd .} \end{cases}$$

The following claim can be applied to show that $\sum_{n=1}^{\infty} a_n$ converges:

"If there are some real $0 < q < 1$ and integer $N \geq 0$ such that $|b_n|^{1/n} \leq q$ for all $n \geq N$, then $\sum_{n=1}^{\infty} b_n$ converges."
Choose the q to complete the proof.
(A) $\frac{3}{5}$; (B) $\frac{5}{6}$; (C) $\frac{1}{2}$; (D) $\frac{1}{3}$.

Solution 20. Since

$$\lim_{n\to\infty} |a_{2n}|^{1/(2n)} = \lim_{n\to\infty} 2^{(-1+1/\sqrt{2n})} = \lim_{n\to\infty} 2^{(-1+0)} = \frac{1}{2},$$
$$\lim_{n\to\infty} |a_{2n+1}|^{1/(2n+1)} = \lim_{n\to\infty} 3^{(-1+1/\sqrt{2n+1})} = \lim_{n\to\infty} 3^{(-1+0)} = \frac{1}{3}.$$

Hence $|a_n| \leq 2^{-1/2}$ for $n \geq 4$. By the claim above, $\sum_{n=1}^{\infty} a_n$ converges. We can see that $|a_n|^{1/n} > \frac{1}{2}$ for all even n. On the other hand, by the limits above, for any $q > \frac{1}{2}$, there is an $M \in \mathbb{N}$ such that for any $n \geq M$,

$$|a_{2n}|^{1/(2n)} < q, \quad \text{and} \qquad |a_{2n+1}|^{1/(2n+1)} < q.$$

Ans: (A)(B).

Question 21. [11.6-P-★] Prove the following claim: If there are some real $0 < q < 1$ and integer $N \geq 0$ such that

$$|b_n|^{1/n} \leq q, \qquad \text{for all } n \geq N,$$

then $\sum_{n=1}^{\infty} b_n$ is convergent.

Solution 21. By the assumption, one has $|b_n| \leq q^n$ for $n \geq N$. Note that $\sum_{n=N}^{\infty} q^n$ is convergent. By the comparison test, $\sum_{n=N}^{\infty} |b_n|$ converges and thus $\sum_{n=1}^{\infty} |b_n|$ converges. As the absolute convergence implies the convergence, $\sum_{n=1}^{\infty} b_n$ is convergent. □

Future Gadget Laboratory, Department of Applied Mathematics, Science Building A, Guangfu Campus, National Chiao Tung University, Hsinchu city, 30010, Taiwan

計算證明題參考答案

- **Answer to Q1.**

(A) First, we apply the method of Lagrange multiplier to find P. Set $g(x,y)=x^{-2}+y^{-2}$.

Note that

$$\underline{\nabla f=-\langle x^{-2},y^{-2}\rangle}\,(\text{得 1 分}),\quad \underline{\nabla g=-2\langle x^{-3},y^{-3}\rangle}\,(\text{得 1 分}).$$

As

$$\underline{\nabla g(x,y)\neq\langle 0,0\rangle \text{ for } (x,y)\in\{g=1\}},\quad (\text{得 1 分})$$

P satisfies

$$\begin{cases}\underline{\nabla f=\lambda\nabla g}, (\text{得 2 分})\\ \underline{g=1}. (\text{得 1 分})\end{cases}$$

Solving the above system gives $\underline{(\sqrt{2},\sqrt{2}) \text{ and } (-\sqrt{2},-\sqrt{2})}$. (得 2 分)

Since $f(\sqrt{2},\sqrt{2})=\sqrt{2}$ and $f(-\sqrt{2},-\sqrt{2})=-\sqrt{2}$, $\underline{P=(\sqrt{2},\sqrt{2})}$. (得 1 分)

Or

Set

$$\begin{cases}\frac{1}{x}=\cos\theta,\\ \frac{1}{y}=\sin\theta,\end{cases}\quad (\text{ 得 2 分})$$

So

$\frac{1}{x}+\frac{1}{y}=\cos\theta+\sin\theta$, $(\theta\neq\frac{n\pi}{2}, n\in\mathbb{Z})$(得 1 分) $\frac{d}{d\theta}(\cos\theta+\sin\theta)=\underline{-\sin\theta+\cos\theta=0}$. (得 1 分)

$\Leftrightarrow \cos\theta=\sin\theta\Leftrightarrow\tan\theta=1$

$\Leftrightarrow \theta=\frac{\pi}{4}+n\pi, n\in\mathbb{Z}$. (得 2 分) $(x,y)=\underline{\pm(\sqrt{2},\sqrt{2})}$ (得 2 分)

Since $f(\sqrt{2},\sqrt{2})=\sqrt{2}$ and $f(-\sqrt{2},-\sqrt{2})=-\sqrt{2}$, $\underline{P=(\sqrt{2},\sqrt{2})}$ (得 1 分)

Or

By the Schwarz Inequality (柯西不等式)

$(\frac{1}{x}+\frac{1}{y})^2\leq(\frac{1}{x^2}+\frac{1}{y^2})(1^2+1^2)$. (得 5 分)

Thus

$-\sqrt{2}\leq\frac{1}{x}+\sqrt{1}y\leq\sqrt{2}$. (得 1 分)

$\frac{1}{x}+\sqrt{1}y=\sqrt{2}\Leftrightarrow\underline{\frac{1}{x}=\frac{1}{y}=t>0}$ (得 2 分)

(B) Next, set $\gamma=\langle t,at^2+bt\rangle$ and $h(t)=f(\gamma(t))=t^{-1}+(at^2+bt)^{-1}$.

Since P sits on the parametric curve $\gamma(t)$, $(\sqrt{2},\sqrt{2})=\gamma(t)$ for some $t\in\mathbb{R}$.

This implies $t=\sqrt{2}$ and $\underline{2a+\sqrt{2}b=\sqrt{2}}$. (得 1 分)

Note that

$$h'(t)=\nabla f(\gamma(t))\cdot\langle 1,2at+b\rangle=-\frac{1}{t^2}-\frac{2at+b}{(at^2+bt)^2}.$$

By Fermat's theorem, as P is a local minimum of f over $\gamma(t)$, one has $h'(\sqrt{2})=0$.

This implies $\underline{2\sqrt{2}a+b=-1}$. (得 1 分)

Solving the following system

$$\begin{cases}2a+\sqrt{2}b=\sqrt{2},\\ 2\sqrt{2}a+b=-1,\end{cases}$$

leads to $\underline{(a,b)=(-\sqrt{2},3)}$. (得 1 分)

- **Answer to Q2.**

(A) Observe that

$$\underline{\lim_{n\to\infty}|a_{2n}|^{1/(2n)}=\tfrac{1}{2}\lim_{n\to\infty}2^{1/\sqrt{2n}}=\tfrac{1}{2}}\quad(\text{得 1 分}),$$

and

$$\underline{\lim_{n\to\infty}|a_{2n+1}|^{1/(2n+1)}=\tfrac{1}{3}\lim_{n\to\infty}3^{1/\sqrt{2n+1}}=\tfrac{1}{3}}\quad(\text{得 1 分}).$$

This implies

$$\lim|a_n|^{1/n} \text{ is divergent.}\quad(\text{得 1 分})$$

(B) By the assumption, one has $|b_n| \le q^n$ for $n \ge N$. (得 1 分)

Note that $\sum_{n=N}^{\infty} q^n$ is convergent. (得 2 分)

By the comparison test (得 2 分), $\sum_{n=N}^{\infty} |b_n|$ converges and thus $\sum_{n=1}^{\infty} |b_n|$ converges. (得 1 分)

As the absolute convergence implies the convergence (得 1 分), $\sum_{n=1}^{\infty} b_n$ is convergent.

(C) Note that, for $n \ge 2$,

$$|a_{2n}|^{1/2n} = 2^{-1+1/\sqrt{2n}} \le 2^{-1/2}, \quad |a_{2n+1}|^{1/(2n+1)} = 3^{-1+1/\sqrt{2n+1}} \le 3^{-1/2}.$$

This implies $|a_n|^{1/n} \le 2^{-1/2}$ for $n \ge 4$. (得 3 分) By part (B), $\sum_{n=1}^{\infty} a_n$ is convergent.