SUGGESTED SOLUTION FOR COMPREHENSIVE ASSESSMENT OF CALCULUS (II) IN WINTER, 2020 (108-1)

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ABSTRACT. This text is a document of suggested solution for comprehensive assessment of Calculus (I), which is held on January 8th, 2020. It happened during the first semester on 108th academical year.

Question 1. [03.6-S-★★] The limit $\lim_{x\to 0}(\cos x - \sin x)^{1/\tan x}$ is
(A) 0. (B) 1. (C) e. (D) e^{-1}.

Solution 1. Since $\lim_{x\to 0}\frac{\cos x - 1}{x^2} = -\frac{1}{2}$, $\cos x \approx 1 - \frac{x^2}{2}$ near 0. Similarly, $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1$, $\sin x \approx \tan x \approx x$ near 0. Therefore,

$$\lim_{x\to 0}(\cos x - \sin x)^{1/\tan x} = \lim_{x\to 0}\left(1 - x - \frac{x^2}{2}\right)^{1/x} = \lim_{x\to 0}(1-x)^{1/x} = e^{-1}.$$

Ans: (D).

Question 2. [06.3-S-★★] Let R be the region enclosed by $y^2 - y + x = 0$ and $x = 0$. The **volume** of the solid obtained by rotating R about the line $y = 2$ equals
(A) $\pi/4$. (B) $\pi/2$. (C) π. (D) $3\pi/2$.

Solution 2. Note that $x = -y^2 + y \geq 0 \iff 0 \leq y \leq 1$. By shell method,

$$V = \int_0^1 2\pi|2-y|(-y^2+y)\,dy = 2\pi\int_0^1 y^3 - 3y^2 + 2y\,dy = 2\pi\left(\frac{1}{4}y^4 - y^3 + y^2\right)\Big|_{y=0}^{1} = \frac{\pi}{2}.$$

Ans: (B).

Question 3. [02.5-S-★★★] Consider the following functions.

$$f(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}, \qquad g(x) = \begin{cases} x\cos x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Which of the following statements is **TRUE**?
(A) $\lim_{x\to 0} f(x) = 1$. (B) $\lim_{x\to 0} g(x) = 1$. (C) $\lim_{x\to 0} g(f(x)) = \cos 1$. (D) $\lim_{x\to 0} f(g(x)) = 0$.

Solution 3. (A) $\lim_{x\to 0} f(x) = \lim_{x\to 0} 0 = 0$.
(B) $\lim_{x\to 0} g(x) = \lim_{x\to 0} x\cos x = 0\cdot\cos 0 = 0$.
(C)

$$g(f(x)) = \begin{cases} g(0), & x \neq 0, \\ g(1), & x = 0. \end{cases} = \begin{cases} 1, & x \neq 0, \\ \cos 1, & x = 0. \end{cases}$$

Thus $\lim_{x\to 0} g(f(x)) = \lim_{x\to 0} 1 = 1$.

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(D) Notice that $x\cos x = 0 \Leftrightarrow x = 0$ or $\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. Now,

$$f(g(x)) = \begin{cases} f(x\cos x), & x \neq 0, \\ f(1) \quad , & x = 0. \end{cases} = \begin{cases} 1, & x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence $\lim_{x\to 0} f(g(x)) = \lim_{x\to 0} 0 = 0$.

Ans: (D).

Question 4. [08.1-S-★★] The **length** of the curve $y = \ln(\cos x)$ from $(0, 0)$ to $\left(\frac{\pi}{4}, -\frac{1}{2}\ln 2\right)$ equals

(A) $\ln\left(\sqrt{2} + 1\right)$. (B) $\ln\left(2\sqrt{2} + 1\right)$. (C) $\ln\left(\frac{1}{\sqrt{2}} + 1\right)$. (D) 1.

Solution 4.

$$L = \int ds = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} |\sec x|\, dx = \ln|\sec x + \tan x|\Big|_{x=0}^{\pi/4} = \ln\left(\sqrt{2} + 1\right).$$

Ans: (A).

Question 5. [07.1-S-★] Let f be an odd function defined on $(-\infty, \infty)$. If f' is continuous and $f(1) = 1$, then $\int_0^2 x f'(1 - x)\, dx$ equals

(A) 1. (B) −1. (C) 2. (D) −2.

Solution 5. By integration by parts, chain rule and change of variable,

$$I := \int_0^2 x f'(1 - x)\, dx = x \cdot (-f(1 - x))\Big|_{x=0}^{2} - \int_0^2 (-f(1 - x))\, dx$$

$$= 2 \cdot (-f(-1)) + \int_{-1}^{1} f(u)\, du \quad (\text{set } u = 1 - x).$$

Since f is odd on $(-\infty, \infty)$, $-f(-1) = f(1) = 1$ and $\int_{-1}^{1} f(u)\, du = 0$, we have $I = 2 \cdot 1 + 0 = 2$.

Ans: (C).

Question 6. [04.2-S-★★★★] Let f be a differentiable function defined on $(0, \infty)$. Which of the following must be **TRUE**?

(A) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} f'(x) > 0$. (B) If $\lim_{x\to\infty} f'(x) > 0$, then $\lim_{x\to\infty} f(x) = \infty$.
(C) If $\lim_{x\to 0^+} f(x) = \infty$, then $\lim_{x\to 0^+} f'(x) = -\infty$. (D) If $\lim_{x\to 0^+} f'(x) = -\infty$, then $\lim_{x\to 0^+} f(x) = \infty$.

Solution 6. (A) For $f(x) = \ln x$ on $(-\infty, \infty)$, $\lim_{x\to\infty} f(x) = \infty$, and $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{1}{x} = 0$.

(B) Since $\lim_{x\to\infty} f'(x) := L > 0$, for fixed $0 < m < L$, we can find $t_m > 0$ such that $f'(x) > m$ for $x > t_m$. Hence by mean value theorem, for $x > t_m$, there exists a $c_m \in (t_m, x)$ such that

$$f(x) = f(t_m) + f'(c_m)(x - t_m) > f(t_m) + m(x - t_m) \to \infty \qquad \text{as } x \to \infty.$$

(C) For $f(x) = \frac{1}{x} + \sin\frac{1}{x}$ on $(0, \infty)$. Since $\frac{1}{x} - 1 \leq f(x) \leq \frac{1}{x} + 1$ for $x > 0$, and $\lim_{x\to 0^+} \frac{1}{x} = \infty$. By squeeze theorem, $\lim_{x\to 0^+} f(x) = \infty$.

Now, $f'(x) = -\frac{1}{x^2}\left(\cos\frac{1}{x} + 1\right)$ for $x > 0$.

Set $x_n = \dfrac{1}{(2n-1)\pi}$ for $n \in \mathbb{N}$. Then $x_n > 0$, and $\lim_{n\to\infty} x_n = 0$. But $f'(x_n) = 0$ for every $n \in \mathbb{N}$. Thus $\lim_{x\to 0+} f'(x) \neq -\infty$ (actually, it does not exist).

(D) For $f(x) = -\sqrt{x}$ on $(0,\infty)$, $\lim_{x\to 0+} f(x) = 0$, and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} -\dfrac{1}{2\sqrt{x}} = -\infty$.

Ans: (B).

Question 7. [06.2-S-★★★] Let $V(t)$ be the **volume** obtained by rotating region

$$A(t) := \{(x,y) \mid 0 \le x \le t, 0 \le y \le \frac{1+\sin^2 x}{2}\}$$

about the y-axis. For what value of $-\infty < r < \infty$ and $0 < c < \infty$ does one have

$$\lim_{t\to 0^+} \frac{V(t)}{t^r} = c?$$

(A) $r = 1$, $c = \pi$. (B) $r = 1$, $c = \pi/2$. (C) $r = 2$, $c = \pi$. (D) $r = 2$, $c = \pi/2$.

Solution 7. Notice that $\sin x \approx x$ as x near 0. Thus

$$\begin{aligned} V(t) &= \int_0^t 2\pi xy\ dx = \int_0^t 2\pi x \left(\frac{1+\sin^2 x}{2}\right)\ dx \\ &\approx \int_0^t 2\pi x \left(\frac{1+x^2}{2}\right)\ dx \approx \int_0^t 2\pi x \left(\frac{1}{2}\right)\ dx \text{ (we take lowest term only near 0)} \\ &= \frac{\pi}{2} t^2. \end{aligned}$$

Ans: (D).

Question 8. [05.3-S-★★★] Assume that f is continuous and satisfies the following equation

$$\lim_{n\to\infty} \frac{f\left(\frac{x^2}{n}\right) + f\left(\frac{2x^2}{n}\right) + \cdots f\left(\frac{(n-1)x^2}{n}\right) + f\left(\frac{nx^2}{n}\right)}{n} = \frac{\sin(\pi x)}{x},$$

for every real number $x \neq 0$. Find $f(1)$.

(A) π. (B) $\pi/2$. (C) 0. (D) $-\pi/2$.

Solution 8.

$$L = \lim_{n\to\infty} \frac{1}{x^2} \sum_{k=1}^{n} \left(\frac{x^2}{n}\right) f\left(\frac{kx^2}{n}\right) = \frac{1}{x^2} \int_0^{x^2} f(t)\ dt = \frac{\sin(\pi x)}{x}.$$

Thus $\int_0^{x^2} f(t)\ dt = x\sin(\pi x)$. By fundamental theorem of calculus and chain rule,

$$f(x^2) \cdot 2x = \sin(\pi x) + \pi x \cos(\pi x), \qquad f(1) = \frac{\sin\pi + \pi\cos\pi}{2} = -\frac{\pi}{2}.$$

Ans: (D).

Question 9. [07.1-S-★★] Find the condition under which the value of the following integral must be **positive**

$$\int_0^{2\pi/\beta} e^{\alpha t} \left(\alpha\cos(\beta t) - \beta\sin(\beta t)\right)\ dt.$$

(A) $\alpha > 0$, $\beta < 0$. (B) $\beta > 2\pi\alpha$. (C) $\alpha\beta > 0$. (D) $\beta < 2\pi\alpha$.

Solution 9.

$$\int_0^{2\pi/\beta} e^{\alpha t} \left(\alpha\cos(\beta t) - \beta\sin(\beta t)\right)\ dt = e^{\alpha t}\cos(\beta t)\Big|_{t=0}^{2\pi/\beta} = e^{2\alpha\pi/\beta} - 1 > 0 \ \Leftrightarrow\ \frac{2\alpha\pi}{\beta} > 0 \ \Leftrightarrow\ \alpha\beta > 0.$$

Ans: (C).

Question 10. [04.1-S-★★★★] Consider the function $F(x) = (f \circ g')(x)$, where

$$g(x) = \begin{cases} x^2 - 3 \quad , & \text{if } x \le 1, \\ -\frac{2}{x} - \frac{x-1}{2}, & \text{if } x > 1; \end{cases} \qquad f(x) = 4x^3 - 15x^2 + 12x.$$

Which of the following about the absolute maximum value y_M and absolute minimum value y_m of F on $\{x:\ 0 \le x \le 2 \text{ and } F(x) \text{ is defeind}\}$ is true?

(A) $y_M = 1$, $y_m = 0$. (B) $y_M = \dfrac{11}{4}$, $y_m = -4$.

(C) $y_M = \dfrac{11}{4}$, no absolute minimum value. (D) $y_M = 1$, $y_m = -4$.

Solution 10. Note that $f'(x) = 6(2x-1)(x-2)$,

$$g'(x) = \begin{cases} 2x \quad , & \text{if } x < 1, \\ \frac{2}{x^2} - \frac{1}{2}, & \text{if } x > 1. \end{cases}$$

Notice that g is not differentiable at $x = 1$. And

$$g''(x) = \begin{cases} 2 \ , & \text{if } x < 1, \\ -\frac{4}{x^3}, & \text{if } x > 1. \end{cases}$$

Therefore,

$$F'(x) = (f \circ g')'(x) = f'(g'(x))g''(x) = \begin{cases} 6(4x-1)(2x-2)\cdot 2 \quad , & \text{if } x < 1, \\ 6\left(\frac{4}{x^2} - 2\right)\left(\frac{2}{x^2} - \frac{5}{2}\right)\left(-\frac{4}{x^3}\right), & \text{if } x > 1. \end{cases}$$

To find critical points of F,

$$\begin{aligned} &F'(x) = (f \circ g')'(x) = f'(g'(x))g''(x) = 0 \\ \Leftrightarrow\ & \big((4x-1)(2x-2) = 0 \text{ and } 0 \le x < 1\big) \text{ or } \left(\left(\frac{4}{x^2} - 2\right)\left(\frac{2}{x^2} - \frac{5}{2}\right)\left(-\frac{4}{x^3}\right) = 0 \text{ and } 1 < x \le 2\right) \\ \Leftrightarrow\ & x = \frac{1}{4}, \sqrt{2}. \end{aligned}$$

x	0	2	$\frac{1}{4}$	$\sqrt{2}$
$g'(x)$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$F(x)$	0	0	$\frac{11}{4}$	$\frac{11}{4}$

Set $D = \{x:\ 0 \le x \le 2 \text{ and } F(x) \text{ is defeind}\}$. Since the range of $g'(x)$ on D is

$$\{2x \mid 0 \le x < 1\} \cup \left\{\frac{2}{x^2} - \frac{1}{2} \mid 1 < x \le 2\right\} = [0, 2).$$

Now, from f' and mean value theorem, f is increasing on $(-\infty, 1/2]$ and $[2, \infty)$, and decreasing on $[1/2, 2]$. For $x \in D$, $g'(x) \in [0, 2)$, $\{f(g'(x)) \mid x \in D\} = \{f(y) \mid y \in [0, 2)\} = \left(-4, \dfrac{11}{4}\right]$. Thus $f \circ g'$ has no absolute minimum on D.

`Ans: (C).`

Question 11. [02.3-M-★] Which of the following statements must be **TRUE**?

(A) If $\lim_{x\to 0} |f(x)| = |L|$, then $\lim_{x\to 0} f(x) = L$.

(B) If $\lim_{x\to 0} f(x) = L$, then $\lim_{x\to 0} |f(x)| = |L|$.

(C) If f is an odd function defined in $(-\infty, \infty)$, then $\lim_{x\to 0} f(x) = 0$.

(D) Let f and g be defined in $(-\infty, \infty)$. If f is an even function and g is an odd function, then both $f \circ g$ and $g \circ f$ are even function.

Solution 11. (A) Set $f(x) = [x] + \frac{1}{2}$. Then $f(x) = -\frac{1}{2}$ for $-1 \le x < 0$, and $f(x) = \frac{1}{2}$ for $0 \le x < 1$, and $\lim_{x\to 0} |f(x)| = \lim_{x\to 0} \frac{1}{2} = 1$. But $\lim_{x\to 0^+} f(x) = \frac{1}{2}$, $\lim_{x\to 0^-} f(x) = -\frac{1}{2}$, f has no limit as $x \to 0$.

(B) Since $||f(x)| - |L|| \le |f(x) - L|$ and $\lim_{x\to 0} |f(x) - L| = 0$. By squeeze theorem, $\lim_{x\to 0} ||f(x)| - |L|| = 0$. Hence $\lim_{x\to 0} |f(x)| = |L|$.

(C) Set

$$f(x) = \begin{cases} 1\,, & \text{if } x > 0, \\ 0\,, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Then f is odd function on $(-\infty, \infty)$. But $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = -1$, f has no limit as $x \to 0$.

(D) For $x \in \mathbb{R}$,

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x),$$
$$(g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x).$$

Ans: (B)(D).

Question 12. [05.3-M-★★] Consider the function $f(x) = \int_0^x |\sin t|\, dt$, where $-\infty < x < \infty$. Which of the following statements are **TRUE**?

(A) f is an increasing function. (B) f is a differentiable function.
(C) f' is a continuous function. (D) f' is a differentiable function.

Solution 12. (A) Since $|\sin t| \ge 0$ on $\mathbb{R}$, $f(x) = \int_0^x |\sin t|\, dt$ increases on $\mathbb{R}$. (Actually, f strictly increases on $\mathbb{R}$.)

(B) Since $|\sin t|$ is continuous on $\mathbb{R}$, by fundamental theorem of calculus, $f'(x) = |\sin x|$.

(C) See (B).

(D) $f'(x) = |\sin x|$ is differentiable on $\mathbb{R}\backslash\{k\pi \mid k \in \mathbb{Z}\}$ only.

Ans: (A)(B)(C).

Question 13. [05.3-M-★] Define $f(x) = \displaystyle\int_1^{x^2} \frac{t^2}{2(t^2+1)}\, dt$. Which of the following statements are **TRUE**?

(A) f is continuous on $(-\infty, \infty)$. (B) f' has a slant asymptote $y = x$.
(C) f is concave upward on $(-\infty, \infty)$. (D) The graph of f' is symmetric about the origin.

Solution 13. (A) Let $g(x) = \displaystyle\int_1^{x} \frac{t^2}{2(t^2+1)}\, dt$. Then g is continuous on $\mathbb{R}$. Thus $f(x) = g(x^2)$ is also.

(B) By chain rule and fundamental theorem of calculus, $f'(x) = g'(x^2) \cdot (x^2)' = \dfrac{x^4}{2(x^4+1)} \cdot 2x$.

$$\lim_{x\to\infty} f'(x) - x = \lim_{x\to\infty} \frac{-x}{x^4+1} = 0.$$

Hence $y = x$ is a slant asymptote of f'.

(C) $f''(x) = \dfrac{x^8 + 5x^4}{(x^4+1)^2} \ge 0$ on $\mathbb{R}$. Therefore, f is concave upward on $\mathbb{R}$.

(D) $f'(x) = \dfrac{x^5}{x^4+1}$ on $\mathbb{R}$. $f'(-x) = -f'(x)$ on $\mathbb{R}$, thus $f'$is odd on $\mathbb{R}$.

Ans: (A)(B)(C)(D).

Question 14. [04.3-M-★★★★] Consider the following function:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{for } x \ne 0, \\ \quad 0 \quad , & \text{for } x = 0. \end{cases}$$

Which of the following statements are **TRUE**?

(A) f is differentiable at 0. (B) f has infinitely many local maxima.

(C) The curve $y = f(x)$ has infinitely many inflection points. (D) The curve $y = f(x)$ has a slant asymptote $y = x$.

Solution 14. (A) $\frac{f(x)-f(0)}{x-0} = x\sin\frac{1}{x}$ for $x \neq 0$. Since $-|x| \leq \left|x\sin\frac{1}{x}\right| \leq |x|$ for $x \neq 0$, and $\lim_{x\to 0} -|x| = \lim_{x\to 0}|x| = 0$. By squeeze theorem, $f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0$.

(B) Since $f \in C(\mathbb{R})$, $f\left(\frac{1}{n\pi}\right) = 0$ for all $n \in \mathbb{N}$, and $f(x) > 0$ for all $x \in \left(\frac{1}{(2n+1)\pi}, \frac{1}{2n\pi}\right)$ for all $n \in \mathbb{N}$. By extreme value theorem, f has local maximum on $\left(\frac{1}{(2n+1)\pi}, \frac{1}{2n\pi}\right)$ for all $n \in \mathbb{N}$.

(C) $f''(x) = \left(2 - \frac{1}{x^2}\right)\sin\frac{1}{x} - \frac{2}{x}\cos\frac{1}{x}$ for $x \neq 0$. Set $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi+\pi}$ for every $n \in \mathbb{N}$. Then

$$\begin{cases} f''(x_n) & = -4n\pi \qquad < 0, \\ f''(y_n) & = \quad 4n\pi + 2\pi > 0. \end{cases}$$

Since $f \in C^2(\mathbb{R}\backslash\{0\})$, by intermediate value theorem, there is a $z_n \in (y_n, x_n)$ such that $f''(z_n) = 0$. There are infinitely $(z_n, f(z_n))$ are inflection point of f.

(D)

$$\lim_{x\to\infty} f(x) - x = \lim_{x\to\infty} x\left(x\sin\frac{1}{x} - 1\right) = \lim_{\theta\to 0^+}\frac{\left(\frac{\sin\theta}{\theta} - 1\right)}{\theta} \qquad (\text{set } \theta = \frac{1}{x})$$
$$= \lim_{\theta\to 0^+}\frac{\sin\theta - \theta}{\theta^2} = \lim_{\theta\to 0^+}\frac{\cos\theta - 1}{2\theta} = \lim_{\theta\to 0^+}\frac{-\sin\theta}{2} = 0. \qquad (\text{by L'Hôpital's rule})$$

Ans: (A)(B)(C)(D).

Question 15. [05.3-M-★★] Let f and g be functions which are continuous on $[a,b]$ and differentiable on (a,b) where $a < b$. Which of the following conditions can guarantee that f and g are equal on $[a,b]$?

(A) $\lim_{y\to x} f(y) = \lim_{y\to x} g(y)$ for any x on (a,b). (B) $f'(x) = g'(x)$ for any x on (a,b).

(C) $\int_a^x f(t)\ dt = \int_a^x g(s)\ ds$ for any x on (a,b).

(D) $f(x) + \int_a^x f(t)\ dt = g(x) + \int_a^x g(s)\ ds$ for any x on (a,b).

Solution 15. (A) Since f is continuous on $[a,b]$, $f(x) = \lim_{y\to x} f(y) = \lim_{y\to x} g(y) = g(x)$ for any x on (a,b). Furthermore, $f(a) = \lim_{y\to a^+} f(y) = \lim_{y\to a^+} g(y) = g(a)$. Similarly, $f(b) = g(b)$.

(B) If $f'(x) = g'(x)$ for all x on (a,b), then there is a constant $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all x on (a,b).

(C) Note that if $h \in \mathcal{C}([a,b])$ and $\int_a^x h(t)\ dt = 0$ for all $x \in [a,b]$, then $h \equiv 0$ on $[a,b]$. Clearly, $\int_a^a f(t)\ dt = \int_a^a g(t)\ dt = 0$. $\int_a^b f(t)\ dt = \lim_{x\to b^-}\int_a^x f(t)\ dt = \lim_{x\to b^-}\int_a^x g(t)\ dt = \int_a^b g(t)\ dt$. Hence $\int_a^x (f(t) - g(t))\ dt = 0$ for all $x \in [a,b]$, $f(x) - g(x) = 0$ for all $x \in [a,b]$.

(D) $[f(x) - g(x)] + \int_a^x (f(t) - g(t))\ dt = 0$ for all $x \in (a,b)$. Set $h(x) = f(x) - g(x)$ for all $x \in [a,b]$. Then $h(x) = \int_a^x h(t)\ dt$ for all $x \in (a,b)$. Moreover, $h \in \mathcal{C}([a,b])$, $h(a) = \lim_{x\to a^+} h(x) = \lim_{x\to a^+}\int_a^x h(t)\ dt = \int_a^a h(t)\ dt = 0$.

By fundamental theorem of calculus, $h'(x) + h(x) = 0$ for all $x \in [a,b)$. So $e^x h'(x) + e^x h(x) = (e^x h(x))' = 0$, there is a $c \in \mathbb{R}$ such that $e^x h(x) = c$ for all $x \in [a,b)$. Now, $h(a) = ce^{-a} = \int_a^a h(t)\ dt = 0$, $c = 0$. Thus, $f(x) - g(x) = 0$ for all $x \in [a,b)$. Furthermore, $f(b) = \lim_{y\to b^-} f(y) = \lim_{y\to b^-} g(y) = g(b)$.

Ans: (A)(C)(D).

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Q1 the answer:

(a) By the implicit differentiation, we have

$$2x(x+2)+x^2 = 2yy' \text{ or } y' = \frac{x(3x+4)}{2y}, \quad \text{if } y \neq 0. \quad (\mathbf{2pts})^{*1}$$

Therefore, $y' = 0$ implies $x = -4/3$ or $x = 0$. Putting $x = 0$ into the equation of C, we have $y = 0$. Note that C cannot produce a horizontal tangent of C at $(0,0)$ (see Figure 1). Putting $x = -4/3$ into the equation of C, we obtain $y = \pm\frac{4\sqrt{6}}{9}$. Hence, C has horizontal tangents at

$$\left(-\frac{4}{3}, \pm\frac{4\sqrt{6}}{9}\right) \text{ or } \left(-\frac{4}{3}, \pm\frac{4\sqrt{2}}{3\sqrt{3}}\right). \quad (\mathbf{3pts})^{*2}$$

Remark 0.1.

(*1) 若學生沒有使用 the implicit differentiation, 而是考慮 $y = \pm x\sqrt{x+2}$ 並得到正確的 y'，即

$$y' = \pm\left(\sqrt{x+2} + \frac{x}{2\sqrt{x+2}}\right),$$

仍可得到第一部分的 2 分。

(*2) 若學生最後答案為 $(-\frac{4}{3}, \pm\frac{4\sqrt{6}}{9})$ 與 $(0,0)$，即沒有排除 $(0,0)$，則第二部分的 3 分僅得 2 分。

(b) The area enclosed by the loop is

$$2\int_{-2}^{0} -x\sqrt{x+2}dx \text{ or } \int_{-2}^{0} -x\sqrt{x+2} - x\sqrt{x+2}dx. \qquad (\mathbf{2pts})$$

By u-substitution and a careful calculation, the area is

$$-2\int_{0}^{2}(u-2)\sqrt{u}du = -2\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right)\Big|_0^2 = \frac{32}{15}\sqrt{2}. \qquad (\mathbf{3pts})^{*3}$$

Remark 0.2.

(*3) 在已得到第一部分 2 分的前提下，若最後答案錯誤但有正確的 substitution form

$$-2\int_{0}^{2}(u-2)\sqrt{u}du,$$

則仍可得到第二部分 3 分中的 2 分。

Q2 the answer:

(a) Set $x(t) = e^{-t}\sin t$ and $y(t) = e^{-t}\cos t$. Note that

$$x'(t) = e^{-t}(-\sin t + \cos t), \quad y'(t) = e^{-t}(-\sin t - \cos t). \qquad (\textbf{2pts})$$

This implies

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{2}e^{-t}dt, \qquad (\textbf{1pts})$$

and, hence, the length of $\{\gamma(t)|0 \le t < \infty\}$ equals

$$\int ds = \int_0^\infty \sqrt{2}e^{-t}dt = \sqrt{2}. \qquad (\textbf{1pts})$$

(b) The desired surface area equals

$$\int 2\pi|1-x|ds = 2\sqrt{2}\pi \int_0^\infty (1 - e^{-t}\sin t)e^{-t}dt. \qquad (\textbf{2pts})$$

Since $e^{-t}\sin t \ge -1$, one has

$$\int_0^\infty (1 - e^{-t}\sin t)e^{-t}dt \le 2\int_0^\infty e^{-t}dt = 2 < \infty. \qquad (\textbf{2pts})$$

This implies that the surface area is finite.

(c) By the formula of $x'(t)$ and $y'(t)$ (see Part (a)), one has

$$\frac{dy}{dx} = \begin{cases} 1 & \text{at } \gamma(\pi/2) \text{ and } \gamma(3\pi/2), \\ -1 & \text{at } \gamma(\pi) \text{ and } \gamma(2\pi). \end{cases} \qquad (\textbf{4pts})$$

As a result, the desired area equals

$$\frac{|x(\pi/2) - x(3\pi/2)| \times |y(\pi) - y(2\pi)|}{2} = \frac{(e^{-\pi/2} + e^{-3\pi/2})(e^{-\pi} + e^{-2\pi})}{2}$$
$$= \frac{e^{-3\pi/2}(1 + e^{-\pi})^2}{2}. \qquad (\textbf{3pts})^{\#}$$

Remark 0.3.
(#) 面積有另外兩種計算方式。首先，這四條切線分別為

$$L_1 : x - y = e^{-\pi/2}; \quad L_2 : x + y = -e^{-\pi};$$
$$L_3 : x - y = -e^{-3\pi/2}; \quad L_4 : x + y = e^{-2\pi}.$$

因為該四邊形為矩形，面積為 L_1 與 L_3 的距離乘以 L_2 與 L_4 的距離，亦即

$$\frac{e^{-\pi/2} + e^{-3\pi/2}}{\sqrt{2}} \times \frac{e^{-\pi} + e^{-2\pi}}{\sqrt{2}} = \frac{e^{-3\pi/2}(1 + e^{-\pi})^2}{2}.$$

另一種算法是找出四邊形中的任意三個頂點（四個頂點座標如下），例如 P, Q, R。

$$P = \left(\frac{e^{-\pi/2} - e^{-\pi}}{2}, \frac{-e^{-\pi/2} - e^{-\pi}}{2}\right); \quad Q = \left(\frac{-e^{-3\pi/2} - e^{-\pi}}{2}, \frac{e^{-3\pi/2} - e^{-\pi}}{2}\right);$$
$$R = \left(\frac{-e^{-3\pi/2} + e^{-2\pi}}{2}, \frac{e^{-3\pi/2} + e^{-2\pi}}{2}\right); \quad S = \left(\frac{e^{-\pi/2} + e^{-2\pi}}{2}, \frac{-e^{-\pi/2} + e^{-2\pi}}{2}\right).$$

則面積為

$$\overline{PQ} \times \overline{QR} = \frac{e^{-\pi/2} + e^{-3\pi/2}}{\sqrt{2}} \times \frac{e^{-\pi} + e^{-2\pi}}{\sqrt{2}} = \frac{e^{-3\pi/2}(1 + e^{-\pi})^2}{2}.$$

無論用何種方法，有完整呈現面積計算方式者得 2 分（不完全者得 1 分），最後有正確計算出面積者得第 3 分。