SUGGESTED SOLUTION FOR COMPREHENSIVE ASSESSMENT OF CALCULUS (II) IN SUMMER, 2020 (108-2)

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ABSTRACT. This text is a document of suggested solution for comprehensive assessment of Calculus (II), which is held on June 24th, 2020. It happened during the second semester on 108th academical year.

Question 1. [10.4-S-★] If the length of the polar curve $r = 3\theta^2$ with $0 \le \theta \le 2\pi$ is $a\left[(\pi^2+1)^{3/2} - 1\right]$, then a is equal to

(A) 2. (B) 4. (C) 8. (D) 16.

Solution 1. The **length** of the polar curve is

$$L = \int ds = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(3\theta^2)^2 + (6\theta)^2}\, d\theta = \int_0^{2\pi} |3\theta| \sqrt{\theta^2 + 2^2}\, d\theta = \int_0^{2\pi} 3\theta\sqrt{\theta^2+4}\, d\theta.$$

Set $u = \theta^2 + 4$. Then $\frac{1}{2}du = \theta d\theta$. By change of variable,

$$L = \int_{0^2+4}^{(2\pi)^2+4} \sqrt{u} \cdot \frac{3}{2}\, du = u^{3/2}\Big|_{u=4}^{4(\pi^2+1)} = 8\left[\left(\pi^2+1\right)^{3/2} - 1\right].$$

Ans: (C).

Question 2. [11.10-S-★] Which one is the Maclaurin series of $\cos^2 x$?

(A) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$. (B) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!}$. (C) $\frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!}$. (D) $\frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!}$.

Solution 2.

$$\cos^2 x = \frac{1}{2}\left(\cos(2x) + 1\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!} + 1\right).$$

Ans: (D).

Question 3. [14.5-S-★★] Let $w = \ln(x^2 + y^2 + z^2)$ and

$$x = ue^v \sin u, y = ue^v \cos u, z = ue^v.$$

What is the value of $\frac{\partial w}{\partial v}$ at $(u, v) = (6\pi^2, 7)$?

(A) 2. (B) 4. (C) 6. (D) $48\pi^2 e^7$.

Solution 3. [Way I]

$$w = \ln\left[\left(ue^v \sin u\right)^2 + \left(ue^v \cos u\right)^2 + \left(ue^v\right)^2\right] = \ln\left(2u^2e^{2v}\right) = \ln 2 + 2\ln|u| + 2v.$$

Thus $\left.\frac{\partial w}{\partial v}\right|_{(u,v)=(6\pi^2,7)} = 2\,|_{(u,v)=(6\pi^2,7)} = 2.$

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Key words and phrases. polar curves, series in $\mathbb{R}$, limits and differentiability $\mathbb{R}^n$, integration in $\mathbb{R}^n$.

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[Way II] Note that $(x,y,z)\,|_{(u,v)=(6\pi^2,7)} = \left(6\pi^2e^7\sin(6\pi^2), 6\pi^2e^7\cos(6\pi^2), 6\pi^2e^7\right)$. By chain rule,

$$\begin{aligned}\left.\frac{\partial w}{\partial v}\right|_{(u,v)=(6\pi^2,7)} &= \left.\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}\right|_{(x,y,z,u,v)=(6\pi^2e^7\sin(6\pi^2),6\pi^2e^7\cos(6\pi^2),6\pi^2e^7,6\pi^2,7)}\\ &= \left.\frac{2x}{x^2+y^2+z^2}\cdot ue^v\sin u+\frac{2y}{x^2+y^2+z^2}\cdot ue^v\cos u+\frac{2z}{x^2+y^2+z^2}\cdot ue^v\right|_{(u,v)=(6\pi^2e^7,6\pi^2,7)}\\ &= \frac{2}{72\pi^4e^{14}}\cdot 6\pi^2e^7\left(6\pi^2e^7\sin^2(6\pi^2)+6\pi^2e^7\cos^2(6\pi^2)+6\pi^2e^7\right)=2.\end{aligned}$$

From the ways above, we discover that someone may probably get correct answer by first way (Compute it directly) rather than second way (chain rule). It is a special case.

`Ans: (A).`

Question 4. [15.4-S-★] What is the value of the following iterated integral?

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(1+x^2+y^2)^2}\,dxdy$$

(A) $1/2$. (B) π. (C) 2π. (D) ∞.

Solution 4. Apply the polar coordinate on $\mathbb{R}^2$ ($x=r\cos\theta$, $y=r\sin\theta$). Notice that

$$D=\left\{(x,y)\in\mathbb{R}^2 \mid -\infty<x,y<\infty\right\}=\{(r,\theta)\mid 0\le\theta\le 2\pi,\ r\ge 0\}.$$

So the iterated integral

$$I=\int_0^{2\pi}\int_0^{\infty}\frac{1}{(1+(r\cos\theta)^2+(r\sin\theta)^2)^2}r\,drd\theta=\int_0^{2\pi}\int_0^{\infty}\frac{r}{(1+r^2)^2}\,drd\theta=\left(\int_0^{2\pi}d\theta\right)\left(\lim_{t\to\infty}\int_0^t\frac{r}{(1+r^2)^2}\,dr\right).$$

Set $u=r^2$. Then $\frac{1}{2}du=rdr$. By change of variable,

$$I=2\pi\left(\lim_{t\to\infty}\int_{0^2}^{t^2}\frac{1}{(1+u)^2}\cdot\frac{1}{2}\,du\right)=2\pi\left(\lim_{t\to\infty}\left.\frac{-1}{2(1+u)}\right|_0^{t^2}\right)=2\pi\cdot\frac{1}{2}\left(\lim_{t\to\infty}1-\frac{1}{1+t^2}\right)=\pi\cdot(1-0)=\pi.$$

`Ans: (B).`

Question 5. [14.6-S-★★] Consider the following function

$$f(0,0)=0,\quad f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2}\ \text{ for } (x,y)\neq(0,0).$$

Set $g=f_x$. Then, the **directional derivative** of g at $(0,0)$ in the direction of $\mathbf{u}=\left\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\rangle$ is

(A) 0. (B) $\frac{\sqrt{2}}{2}$. (C) $-\frac{\sqrt{2}}{2}$. (D) $\sqrt{2}$.

Solution 5. First, for $(x,y)\neq(0,0)$,

$$g(x,y)=f_x(x,y)=\frac{[y(x^2-y^2)+xy\cdot 2x](x^2+y^2)-xy(x^2-y^2)\cdot 2x}{(x^2+y^2)^2}=\frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}.$$

For $(x,y)=(0,0)$, by definition of partial derivative,

$$g(0,0)=f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0.$$

Therefore, by definition of the directional derivative,

$$D_{\mathbf{u}}g(0,0)=\lim_{h\to 0}\frac{g\left(0+h\frac{\sqrt{2}}{2},0+h\frac{\sqrt{2}}{2}\right)-f(0,0)}{h}=\lim_{h\to 0}\frac{1}{h}\cdot\frac{\sqrt{2}}{2}h\cdot\frac{\left(\frac{\sqrt{2}}{2}h\right)^4+4\left(\frac{\sqrt{2}}{2}h\right)^2\left(\frac{\sqrt{2}}{2}h\right)^2-\left(\frac{\sqrt{2}}{2}h\right)^4}{\left[\left(\frac{\sqrt{2}}{2}h\right)^2+\left(\frac{\sqrt{2}}{2}h\right)^2\right]^2}=\frac{\sqrt{2}}{2}.$$

Actually, f is continuously differentiable at $(0,0)$, but g is not. Furthermore, g is continuous on $\mathbb{R}$, and has directional derivatives with respect to any direction. But

$$D_{\mathbf{u}}g(0,0) = \nabla g(0,0)\cdot\mathbf{u} = \langle g_x(0,0), g_y(0,0)\rangle\cdot\mathbf{u}$$

only for $\mathbf{u} = \langle \pm 1, 0\rangle$ or $\langle 0, \pm 1\rangle$.

`Ans: (B).`

Question 6. [14.6-S-★★★] Let f be a function of two variables with continuous partial derivatives. Assume that the directional derivatives of f at $(0,0)$ in the direction of $\langle 3,4\rangle$ and $\langle 4,-3\rangle$ are respectively 3 and 4. Then, the **directional derivative** of f at $(0,0)$ in the direction of $\langle 1,1\rangle$ is

(A) 5. (B) -5. (C) $\frac{5\sqrt{2}}{2}$. (D) $-\frac{5\sqrt{2}}{2}$.

Solution 6.

v	$\langle 3,4\rangle$	$\langle 4,3\rangle$	$\langle 1,1\rangle$
normalized vector of v	$\mathbf{p} := \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$	$\mathbf{q} := \left\langle \frac{4}{5}, \frac{3}{5}\right\rangle$	$\mathbf{u} := \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

Notice that

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = \frac{7}{5\sqrt{2}}\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle + \frac{1}{5\sqrt{2}}\left\langle \frac{4}{5}, \frac{3}{5}\right\rangle.$$

Since f_x, f_y are continuous at $(0,0)$, f is differentiable at $(0,0)$. Therefore,

$$D_{\mathbf{u}}f(0,0) = \frac{7}{5\sqrt{2}}D_{\mathbf{p}}f(0,0) + \frac{1}{5\sqrt{2}}D_{\mathbf{q}}f(0,0) = \frac{7}{5\sqrt{2}}\cdot 3 + \frac{1}{5\sqrt{2}}\cdot 4 = \frac{5\sqrt{2}}{2}.$$

Actually, we need only condition that f differentiable at $(0,0)$, to get the $D_{\mathbf{u}}f(0,0)$.

`Ans: (C).`

Question 7. [11.1-S-★★★] Assume that there is a number $\delta > 0$ such that for all $|x| < \delta$,

$$\sum_{n=0}^{\infty} 2^{n+1}a_{n+2}x^{n+1} + \sum_{n=1}^{\infty}\frac{2^n}{a_n}x^n = 1 - a_1 + \sum_{n=1}^{\infty} 3\cdot 2^n x^n$$

Then, $\lim_{n\to\infty} a_n$ is equal to

(A) $\frac{1}{2}$. (B) 1. (C) $\frac{3+\sqrt{5}}{2}$. (D) $\frac{1+\sqrt{3}}{2}$.

Solution 7. Let the dummy variable $n = k-1$. Then

$$\sum_{k=1}^{\infty} 2^k a_{k+1}x^k + \sum_{n=1}^{\infty}\frac{2^n}{a_n}x^n = 1 - a_1 + \sum_{n=1}^{\infty} 3\cdot 2^n x^n \;\Rightarrow\; \sum_{n=1}^{\infty}\left(2^n a_{n+1} + \frac{2^n}{a_n}\right)x^n = 1 - a_1 + \sum_{n=1}^{\infty} 3\cdot 2^n x^n.$$

Thus,

$$\begin{cases} 1 - a_1 = 0, \\ 2^n a_{n+1} + \frac{2^n}{a_n} = 3\cdot 2^n,\; n \geq 1. \end{cases} \Rightarrow \begin{cases} a_1 = 1, \\ a_{n+1} = 3 - \frac{1}{a_n},\; n \geq 1. \end{cases}$$

We will show the sequence $\{a_n\}_{n=1}^{\infty}$ converges by *axiom of completeness*, that is, $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above. If $\lim_{n\to\infty} a_n = L \in \mathbb{R}$, then

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 3 - \frac{1}{a_n} = 3 - \frac{1}{\lim_{n\to\infty} a_n} = 3 - \frac{1}{L}.$$

So $L^2 = 3L - 1$, $L = \frac{3\pm\sqrt{5}}{2}$. But $L = \lim_{n\to\infty} a_n \geq a_1 = 1$, $L \neq \frac{3-\sqrt{5}}{2}$. Thus $L = \frac{3+\sqrt{5}}{2}$.

Here, we will claim $1 \leq a_n < a_{n+1} < L$ for all $n \in \mathbb{N}$ by mathematical induction.

Step I: This assertion holds for $n = 1$.

$$a_2 = 3 - \frac{1}{1} = 2 = \frac{3+1}{2} < \frac{3+\sqrt{5}}{2} = L, \qquad \text{and } a_2 > 1 = a_1.$$

Step II: When this assertion holds for $n = k$, it also holds for $n = k+1$. Suppose that $k \in \mathbb{N}$ and $1 \le a_k < a_{k+1} < L$. So

$$0 < \frac{1}{a_{k+1}} < \frac{1}{a_k}, \; a_{k+2} = 3 - \frac{1}{a_{k+1}} > 3 - \frac{1}{a_k} = a_{k+1} \ge 1.$$

On the other hand, from the equality of L,

$$\frac{1}{a_k} > \frac{1}{L}, \; a_{k+2} = 3 - \frac{1}{a_{k+1}} < 3 - \frac{1}{L} = L.$$

By mathematical induction, $1 \le a_n < a_{n+1} < L$ for all $n \in \mathbb{N}$. That is, $\{a_n\}_{n=1}^{\infty}$ is strictly increasing and bounded above by L.

Actually, those series converges for $0 < \delta \le \frac{1}{2}$.

Ans: (C).

Question 8. [14.3-S-★★★★] Let $f(x, y)$ be a continuous function defined for $-\infty < x, y < \infty$ and satisfying

$$\lim_{(x,y)\to(e,\pi)} \frac{f(x,y) - 1 - 2(x-e) - 3(y-\pi)}{\sqrt{(x-e)^2 + (y-\pi)^2}} = 0.$$

Then, $f(e,\pi) + f_x(e,\pi) + f_y(e,\pi)$ is equal to

(A) $e\pi$. (B) 2. (C) 3. (D) 6.

Solution 8. [Way I]

$$\lim_{x\to e} \frac{f(x,\pi) - 1 - 2(x-e) - 3(\pi-\pi)}{\sqrt{(x-e)^2 + (\pi-\pi)^2}} = \lim_{x\to e} \frac{f(x,\pi) - 1 - 2(x-e)}{|x-e|} = 0$$

$$\Rightarrow \lim_{x\to e} \left| \frac{f(x,\pi) - 1 - 2(x-e)}{|x-e|} \right| = \lim_{x\to e} \left| \frac{f(x,\pi) - 1 - 2(x-e)}{x-e} \right| = |0| = 0 \;\Rightarrow\; \lim_{x\to e} \frac{f(x,\pi) - 1 - 2(x-e)}{x-e} = 0.$$

Notice that $\lim\limits_{x\to e} x - e = 0$, we have

$$\lim_{x\to e} f(x,\pi) - 1 - 2(x-e) = \left(\lim_{x\to e} \frac{f(x,\pi) - 1 - 2(x-e)}{x-e} \right) \left(\lim_{x\to e} x - e \right) = 0 \cdot 0 = 0,$$

$\lim\limits_{x\to e} f(x,\pi) = \lim\limits_{x\to e} 1 + 2(x-e) = 1$. Since f is continuous at (e,π), $\lim\limits_{x\to e} f(x,\pi) = f(e,\pi) = 1$. Finally,

$$f_x(e,\pi) = \lim_{x\to e} \frac{f(x,\pi) - f(e,\pi)}{x-e} = \lim_{x\to e} \frac{f(x,\pi) - 1}{x-e}$$
$$= \lim_{x\to e} \frac{(f(x,\pi) - 1 - 2(x-e)) + 2(x-e)}{x-e} = 0 + \lim_{x\to e} \frac{2(x-e)}{x-e} = 2.$$

Similarly, we can show that $f_y(e,\pi) = 3$. So $f(e,\pi) + f_x(e,\pi) + f_y(e,\pi) = 1 + 2 + 3 = 6$.

[Way II] Since f is continuous at (e,π),

$$f(e,\pi) = \lim_{(x,y)\to(e,\pi)} f(x,y) = \lim_{(x,y)\to(e,\pi)} (f(x,\pi) - 1 - 2(x-e) - 3(y-\pi)) + (1 + 2(x-e) + 3(y-\pi))$$
$$= \lim_{(x,y)\to(e,\pi)} \frac{f(x,y) - 1 - 2(x-e) - 3(y-\pi)}{\sqrt{(x-e)^2 + (y-\pi)^2}} \sqrt{(x-e)^2 + (y-\pi)^2} + (1 + 2(x-e) + 3(y-\pi))$$
$$= 0 \cdot \sqrt{(e-e)^2 + (\pi-\pi)^2} + (1 + 2(e-e) + 3(\pi-\pi)) = 1.$$

Let $g(x,y) = 1 + 2(x-e) + 3(y-\pi)$ and $h(x,y) = f(x,y) - g(x,y)$. Then $h(e,\pi) = 0$, and

$$\lim_{(x,y)\to(e,\pi)} \frac{h(x,y) - [h(e,\pi) + \langle 0,0\rangle \cdot (x,y)]}{\sqrt{(x-e)^2 + (y-\pi)^2}} = 0.$$

Hence h is differentiable at (e, π) with $\nabla h(e, \pi) = \langle 0, 0 \rangle$. Since g is differentiable at (e, π) with $\nabla g(e, \pi) = \langle 2, 3 \rangle$, $f = g + h$ is differentiable at (e, π) with $\nabla f(e, \pi) = \nabla g(e, \pi) + \nabla h(e, \pi) = \langle 2, 3 \rangle$. So $f(e, \pi) + f_x(e, \pi) + f_y(e, \pi) = 1 + 2 + 3 = 6$.

`Ans: (D).`

Question 9. [11.8-S-★★★★★] Suppose that $E = \left\{-\frac{1}{3}, \frac{1}{4}, 1, 2\right\}$. How many points $x \in E$ at which the power series $\sum_{n=1}^{\infty} (\sin n)(2x-1)^n$ **converges**?

(A) 1. (B) 2. (C) 3. (D) 4.

Solution 9. Clearly, $|\sin n| \leq 1$ for all $n \in \mathbb{N}$. Thus $|(\sin n)(2x-1)^n| \leq |2x-1|^n$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} |2x-1|^n$ converges $\Leftrightarrow$ $|2x-1| < 1$ $\Leftrightarrow$ $0 < x < 1$, by comparison test, $\sum_{n=1}^{\infty} (\sin n)(2x-1)^n$ converges absolutely for $x = \frac{1}{4}$. Unfortunately, we are not sure the power series diverges for other points in E at the moment.

Now, we will show that $\{\sin n\}_{n=1}^{\infty}$ does not converges to 0. Suppose not, $\lim_{n\to\infty} \sin n = 0$. For every $n \in \mathbb{N}$,

$$\begin{cases} \cos^2 n + \sin^2 n = 1, \\ \sin(n+1) = \sin n \cos 1 + \sin 1 \cos n. \end{cases}$$

Hence

$$\lim_{n\to\infty} \cos n = \lim_{n\to\infty} \frac{\sin(n+1) - \sin n \cos 1}{\sin 1} = \frac{0 - 0 \cdot \cos 1}{\sin 1} = 0.$$

But $\lim_{n\to\infty} \cos^2 n + \sin^2 n = 0^2 + 0^2 = 0 \neq 1$. Contradiction.

Thus for any $x \notin (0, 1)$, $\{(\sin n)(2x-1)^n\}_{n=1}^{\infty}$ does not converges to 0. (If not,

$$\lim_{n\to\infty} \sin n = \lim_{n\to\infty} \frac{(\sin n)(2x-1)^n}{(2x-1)^n} = \left(\lim_{n\to\infty} (\sin n)(2x-1)^n\right)\left(\lim_{n\to\infty} \frac{1}{(2x-1)^n}\right) = 0,$$

Contradiction.) By divergence test, $\sum_{n=1}^{\infty} (\sin n)(2x-1)^n$ diverges.

`Ans: (A).`

Question 10. [15.6-S-★★] For $t > 0$, let $A(t)$ be the area of the surface $f(x, y) = xy\cos(txy)$ with $x^2 + y^2 \leq 3$. Then $\lim_{t\to 0^+} A(t)$ is equal to

(A) $\frac{7\pi}{3}$. (B) $\frac{14\pi}{3}$. (C) $\frac{21\pi}{3}$. (D) $\frac{28\pi}{3}$.

Solution 10.

$$\begin{cases} f_x(x, y) = y\cos(txy) - txy^2 \sin(txy) \to y \cdot \cos 0 - 0 = y, \\ f_y(x, y) = x\cos(txy) - tx^2 y \sin(txy) \to x \cdot \cos 0 - 0 = x, \end{cases} \quad \text{as } t \to 0^+.$$

By the polar coordinate on $\mathbb{R}^2$ $(x = r\cos\theta,\ y = r\sin\theta)$, we have

$$D = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 3\right\} = \left\{(r, \theta) \mid 0 \leq \theta \leq 2\pi,\ r \geq \sqrt{3}\right\}.$$

So the limit is equal to

$$\lim_{t\to 0^+} A(t) = \lim_{t\to 0^+} \int_D \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}\, dxdy$$

$$= \int_D \left(\lim_{t\to 0^+} \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}\right) dxdy$$

$$= \int_D \sqrt{1 + y^2 + x^2}\, dxdy = \int_0^{\sqrt{3}} \int_0^{2\pi} \sqrt{1 + r^2}\, r d\theta dr = 2\pi \int_0^{\sqrt{3}} \sqrt{1 + r^2}\, r dr$$

Set $u = 1 + r^2$. Then $\frac{1}{2}\ du = rdr$. By change of variable,

$$\lim_{t\to 0^+} A(t) = 2\pi \int_1^4 \frac{1}{2}\sqrt{u}\ du = \frac{2\pi}{3}u^{3/2}\Big|_1^4 = \frac{14\pi}{3}.$$

`Ans: (B).`

Question 11. [10.4-M-★★★] Consider the following polar equations.

$$\gamma_1:\ r^2 = \sin(2\theta),\quad \gamma_2:\ r^2 = \cos(2\theta),\quad \gamma_3:\ r = 1 + \sin\theta,\quad \gamma_4:\ r = -1 + \sin\theta.$$

Which of the following statements are **TRUE**?

(A) There are three horizontal tangent lines on the graph of γ_1.

(B) The graph of γ_2 is symmetric about the x-axis and the y-axis.

(C) The graph of γ_3 and γ_4 are different.

(D) The area of the region enclosed by the graph of γ_1 is the same as the area of the region enclosed by the graph of γ_2.

Solution 11. (A) Notice that the slope of tangent line at $\left(r\cos\tilde{\theta}, r\sin\tilde{\theta}\right)$ in γ_1 is

$$\begin{aligned}
&\frac{dy}{dx}\Big|_{(x,y)=\left(r\cos\tilde{\theta}, r\sin\tilde{\theta}\right)} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}\Big|_{(r,\theta)=\left(r(\tilde{\theta}),\tilde{\theta}\right)} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}\Big|_{(r,\theta)=\left(r(\tilde{\theta}),\tilde{\theta}\right)} \\
&= \frac{(r^2)'\sin\theta + 2r^2\cos\theta}{2rr'\cos\theta - 2r^2\sin\theta}\Big|_{(r,\theta)=\left(r(\tilde{\theta}),\tilde{\theta}\right)} = \frac{(r^2)'\sin\theta + 2r^2\cos\theta}{2rr'\cos\theta - 2r^2\sin\theta}\Big|_{(r,\theta)=\left(r(\tilde{\theta}),\tilde{\theta}\right)} \\
&= \frac{2\cos(2\theta)\sin\theta + 2\sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta - 2\sin(2\theta)\sin\theta}\Big|_{(r,\theta)=\left(r(\tilde{\theta}),\tilde{\theta}\right)} = \frac{\sin(3\theta)}{\cos(3\theta)}\Big|_{(r,\theta)=\left(r(\tilde{\theta}),\tilde{\theta}\right)} \\
&= \tan(3\theta)\,|_{(r,\theta)=\left(r(\tilde{\theta}),\tilde{\theta}\right)} = \tan(3\tilde{\theta}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{dy}{dx}\Big|_{(x,y)=\left(r\cos\tilde{\theta}, r\sin\tilde{\theta}\right)} = 0 \iff \begin{cases}\tan(3\tilde{\theta}) = 0, \\ r^2 = \sin(2\tilde{\theta}) \geq 0\end{cases} \\
&\iff \begin{cases}\tilde{\theta} = \frac{m\pi}{3},\ m \in \mathbb{Z}; \\ n\pi \leq \tilde{\theta} \leq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}\end{cases} \iff \left(r\cos\tilde{\theta}, r\sin\tilde{\theta}\right) = (0,0), \left(\frac{3^{1/4}}{2\sqrt{2}}, \pm\frac{3^{3/4}}{2\sqrt{2}}\right).
\end{aligned}$$

In summary, $y = 0$, $y = \frac{3^{3/4}}{2\sqrt{2}}$ and $y = -\frac{3^{3/4}}{2\sqrt{2}}$ are horizontal tangent lines in total.

(B) Notice that $(x, y) = (r\cos\theta, r\sin\theta)$ locates on the polar curve γ_2 if and only if $n\pi - \frac{\pi}{4} \leq \theta \leq n\pi + \frac{\pi}{4}$, $n \in \mathbb{Z}$. Now,

$$\begin{cases}(x, -y) = (r\cos\theta, -r\sin\theta) = (r\cos(-\theta), r\sin(-\theta)) \in \gamma_2, \\ (-x, y) = -(x, -y) = -(r\cos(-\theta), r\sin(-\theta)) = ((-r)\cos(-\theta), (-r)\sin(-\theta)) \in \gamma_2.\end{cases}$$

Therefore, the graph of γ_2 is symmetric about the x-axis and the y-axis.

(C) Notice that $(x, y) = (r\cos\theta, r\sin\theta)$ locates on the polar curve γ_3 if and only if

$$\begin{aligned}
&(r\cos\theta, r\sin\theta) = (\cos\theta + \cos\theta\sin\theta, \sin\theta + \sin^2\theta) = (-(-\cos\theta) + (-\cos\theta)(-\sin\theta), -(-\sin\theta) + (-\sin\theta)^2) \\
&= (-\cos(\theta+\pi) + \cos(\theta+\pi)\sin(\theta+\pi), -\sin(\theta+\pi) + \sin^2(\theta+\pi)) \in \gamma_4
\end{aligned}$$

Therefore, the graph of γ_3 and γ_4 are coincide.

(D) Notice that

$$\sin(2\theta) = \cos\left(\frac{\pi}{2} - 2\theta\right) = \cos\left(2\left(\frac{\pi}{4} - \theta\right)\right).$$

Thus, we can reflect the graph of γ_1 about the line, $\theta = \dfrac{\pi}{8}$, to get γ_2. Hence, the area of the region enclosed by the graph of γ_1 is the same as the area of the region enclosed by the graph of γ_2.

Ans: (A)(B)(D).

Question 12. [11.4-M-★★] Which of the following series are **convergent**?

(A) $\displaystyle\sum_{n=1}^{\infty} n\tan\frac{1}{n}$. (B) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$. (C) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$. (D) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

Solution 12. (A) Notice that

$$\lim_{x\to 0} \frac{\tan x}{x} = \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right) = \frac{1}{1}\cdot 1 = 1.$$

Since $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$, we have $\displaystyle\lim_{n\to\infty} n\tan\frac{1}{n} = \lim_{n\to\infty}\frac{\tan(1/n)}{1/n} = 1 \neq 0$. By divergent test, this series diverges.

(B) For $x \geq 2$, $x^2 \geq 2x > 0$, and $e^{x^2} > e^{2x} = e^x \cdot e^x$ and $e^x > x$. So for $x \geq 2$,

$$0 < \frac{x}{e^{x^2}} \leq \frac{x}{e^x\cdot e^x} = \frac{1}{e^x}\cdot\frac{x}{e^x} < \left(\frac{1}{e}\right)^x.$$

Since $0 < \dfrac{1}{e} < 1$, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n$ converges. By comparison test, $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converges. Moreover, $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2} \leq \frac{1}{e-1}$.

(C) For $n \in \mathbb{N}$, $0 < \sqrt{n} < n$, and so

$$\frac{1}{\sqrt{n}(\sqrt{n}+1)} = \frac{1}{n+\sqrt{n}} > \frac{1}{n+n} = \frac{1}{2n} > 0.$$

By p-series text, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges. By comparison test, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}(\sqrt{n}+1)}$ diverges.

(D) Notice that for $n \in \mathbb{N}$,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \geq \binom{n}{0}\cdot 1 + \binom{n}{1}\cdot\frac{1}{n} = 1+1 = 2.$$

So

$$0 < \left(\frac{n}{n+1}\right)^{n^2} = \left(\frac{1}{(1+\frac{1}{n})^n}\right)^n \leq \left(\frac{1}{2}\right)^n.$$

Since $0 < \dfrac{1}{2} < 1$, the geometry series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges. By comparison test, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges. Moreover, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2} \leq 1$.

Ans: (B)(D).

Question 13. [14.6-M-★★★] Consider the following function

$$f(0,0) = 0, \quad f(x,y) = \frac{4xy^2}{4x^2+y^4} \text{ for } (x,y) \neq (0,0).$$

Which of the following statements are **TRUE**?

(A) f is continuous at $(0,0)$. (B) $D_{\mathbf{u}}f(0,0)$ exists for any unit vector $\mathbf{u}$.

(C) $(0,0)$ is a critical point of f. (D) f_x and f_y are continuous at $(0,0)$.

Solution 13. (A) Let $x = y^2$. Then $\lim_{y\to 0}(x, y) = \lim_{y\to 0}(y^2, y) = (0, 0)$, and

$$f(x, y) = f(y^2, y) = \frac{4(y^2)y^2}{4(y^2)^2 + y^4} = \frac{4}{5} \to \frac{4}{5} \neq f(0, 0) \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } x = y^2.$$

Hence f is discontinuous at $(0, 0)$.

(B) Let $\theta \in \mathbb{R}$ and $\mathbf{u} = (\cos\theta, \sin\theta)$.

$$D_{\mathbf{u}}f(0,0) = \lim_{t\to 0}\frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} = \lim_{t\to 0}\frac{1}{t}\frac{4t^3\cos\theta\sin^2\theta}{4t^2\cos^2\theta + t^4\sin^4\theta}$$
$$= \begin{cases} \lim_{t\to 0}\dfrac{4\cos\theta\sin^2\theta}{4\cos^2\theta + t^2\sin^4\theta} = \dfrac{\sin^2\theta}{\cos\theta}, & \text{if } \cos\theta \neq 0; \\ \lim_{t\to 0} 0 = 0, & \text{if } \cos\theta = 0. \end{cases}$$

(C) From (B), $f_x(0,0) = D_{\langle 1,0\rangle}f(0,0) = 0$, and $f_y(0,0) = D_{\langle 0,1\rangle}f(0,0) = 0$. So $(0,0)$ is a critical point of f.

(D) For $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{4y^2(4x^2 + y^4) - 4xy^2\cdot 8x}{(4x^2 + y^4)^2} = \frac{4y^2(-4x^2 + y^4)}{(4x^2 + y^4)^2};$$
$$f_y(x, y) = \frac{8xy(4x^2 + y^4) - 4xy^2\cdot 4y^3}{(4x^2 + y^4)^2} = \frac{8xy(4x^2 - y^4)}{(4x^2 + y^4)^2}.$$

Let $x = y^2$. Then $\lim_{y\to 0}(x, y) = \lim_{y\to 0}(y^2, y) = (0, 0)$, and

$$f_x(x, y) = f_x(y^2, y) = \frac{4y^2(-4(y^2)^2 + y^4)}{(4(y^2)^2 + y^4)^2} = \frac{-12y^6}{25y^8} \text{ has no limit} \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } x = y^2.$$
$$f_y(x, y) = f_y(y^2, y) = \frac{8y^2\cdot y(4(y^2)^2 - y^4)}{(4(y^2)^2 + y^4)^2} = \frac{24y^7}{25y^8} \text{ has no limit} \quad \text{as} \quad (x, y) \to (0, 0) \text{ along } x = y^2.$$

Ans: (B)(C).

Question 14. [15.3-M-★★] Which of the following statements are **TRUE**?

(A)
$$\int_3^6\int_1^3 \frac{x-1}{y-2}\sin(x-y)\ dxdy = \int_1^3\int_3^6 \frac{x-1}{y-2}\sin(x-y)\ dydx.$$

(B)
$$\int_0^1\int_0^x \sqrt{x^2+y}\ dydx = \int_0^x\int_0^1 \sqrt{x^2+y}\ dxdy \text{ for all } x.$$

(C)
$$\int_{-2}^2\int_0^5 e^{-2x^2-2y^2}\sin y\cos x\ dxdy = 0.$$

(D)
$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} e^{-z^2}\ dzdydx = \int_0^1\int_0^{1-z}\int_0^{y^2} e^{-z^2}\ dxdydz.$$

Solution 14. (A) Let $E = [1,3]\times[3,6]$ and $f(x, y) = \dfrac{x-1}{y-2}\sin(x-y)$ for $(x, y) \in E$. Then f is continuous on E. By Fubini's theorem, it is true.

(B)
$$\int_0^1\int_0^x \sqrt{x^2+y}\ dydx = \int_0^1\left(\frac{2}{3}\left(x^2+y\right)^{3/2}\Big|_{y=0}^{x}\right)\ dx = \frac{2}{3}\int_0^1\left(\left(x^2+x\right)^{3/2} - x^3\right)\ dx$$

$$\int_0^x \int_0^1 \sqrt{x^2+y}\ dxdy = \int_0^x \left(\frac{1}{2}\left(x\sqrt{x^2+y} + y\ln\left|x+\sqrt{x^2+y}\right|\right)\Big|_{x=0}^{1} \right)\ dy$$
$$= \int_0^x \frac{1}{2}\left(\sqrt{y+1} + y\ln\left(1+\sqrt{y+1}\right) - \frac{1}{2}y\ln|y|\right)\ dy$$

Clearly, the left integral is a constant, and the right one varies with respect to x. Thus they are not identical.

(C) By Fubini's theorem, the integral is equal to

$$\left(\int_0^5 e^{-2x^2}\cos x\ dx\right)\left(\int_{-2}^2 e^{-2y^2}\sin y\ dy\right).$$

It is obviously that e^{-2y^2} and $\sin y$ are even function and odd function, respectively. We have $e^{-2y^2}\sin y$ is an odd function,

$$\left(\int_0^5 e^{-2x^2}\cos x\ dx\right)\left(\int_{-2}^2 e^{-2y^2}\sin y\ dy\right) = \left(\int_0^5 e^{-2x^2}\cos x\ dx\right)\cdot 0 = 0.$$

(D) Notice that

$$\begin{cases} 0 \le z \le 1-y, \\ \sqrt{x} \le y \le 1, \\ 0 \le x \le 1. \end{cases} \Leftrightarrow \begin{cases} 0 \le z, \\ 0 \le y \le 1-z \le 1, \\ x \le y^2 \le 1, \\ 0 \le x \le 1. \end{cases} \Leftrightarrow \begin{cases} 0 \le x \le y^2, \\ 0 \le y \le 1-z, \\ 0 \le z \le 1. \end{cases}$$

By Fubini's theorem, these integrals coincide.

`Ans: (A)(C)(D).`

Question 15. [14.8-M-★★] Subject to which constraint, the function $f(x,y,z) = x^2+y^2-z^2$ does **NOT** attain its maximum value?

(A) $x+y+z=1$. (B) $x^2+y^2+z^2=1$. (C) $e^{xz}+y^2=1$. (D) $e^{x^3+z^3+x+z}=y^2$.

Solution 15. At beginning, we need to announce that: all the constraint below are **closed**.

(A) For any $t\in\mathbb{R}$, $(1-t,t,0)$ locates on the constraint $x+y+z=1$, and $f(1-t,t,0)=(1-t)^2+t^2-0^2\to\infty$ as t approaches ∞.

(B) $|f(x,y,z)| \le x^2+y^2+z^2 \le 1$.

(C) For any $t\in\mathbb{R}$, $(t,0,0)$ locates on the constraint $e^{xz}+y^2=1$, and $f(t,0,0)=t^2\to\infty$ as t approaches ∞.

(D) For any $t\in\mathbb{R}$, $(t,\sqrt{e^{t^3+t}},0)$ locates on the constraint $e^{x^3+z^3+x+z}=y^2$, and the limit $f(t,\sqrt{e^{t^3+t}},0) = t^2+e^{t^3+t}\to\infty$ as t approaches ∞.

`Ans: (A)(C)(D).`

Question 16. [14.5-S-★] Let $f(x,y) = \displaystyle\int_{x-y}^{x+y} e^{t^2}\ dt$. Then, $f_{xy}(1,1) =$

(A) $2e^2$. (B) $4e^2$. (C) $2e^4$. (D) $4e^4$.

Solution 16. Let $g:\mathbb{R}\to\mathbb{R}$ such that $g(v) = \displaystyle\int_0^v e^{t^2}\ dt$. By fundamental theorem of calculus, $g'(v)=e^{v^2}$. Now, $f(x,y)=g(x+y)-g(x-y)$. By chain rule in several variables,

$$f_x(x,y) = g'(x+y)\cdot\frac{\partial}{\partial x}(x+y) + g'(x-y)\cdot\frac{\partial}{\partial x}(x-y) = e^{(x+y)^2}+e^{(x-y)^2},$$

and thus

$$f_{xy}(x,y) = \left(\frac{d}{dt}e^{t^2}\Big|_{t=x+y}\right)\cdot\frac{\partial}{\partial x}(x+y)^2 + \left(\frac{d}{dt}e^{t^2}\Big|_{t=x-y}\right)\cdot\frac{\partial}{\partial x}(x-y)^2 = 2(x+y)e^{(x+y)^2}+2(x-y)e^{(x-y)^2}.$$

Thus $f_{xy}(1,1) = 2\cdot(1+1)e^{2^2}+0 = 4e^4$.

`Ans: (D).`

Question 17. [14.8-M-★] Let α and β be the absolute extreme values of $f(x,y,z) = xy + z^2$ subject to the constraints $x - y = 0$ and $x^2 + y^2 + z^2 \leq 4$. Which of the following values belongs to $\{\alpha, \beta\}$?

(A) 4. (B) 0. (C) 2. (D) 1.

Solution 17. [Way I] By constraint $x - y = 0$, $x = y$, and so $f(x,y,z) = xy + z^2 = x^2 + z^2 \geq 0$. Since $(0,0,0)$ locates constraints and $f(0,0,0) = 0$, the minimum is 0.

On the other hand, $f(x,y,z) = x^2 + z^2 = x^2 + y^2 + z^2 - y^2 \leq 4 - y^2 \leq 4$. Since $(0,0,2)$ locates constraints and $f(0,0,2) = 4$, the maximum is 4.

[Way II] $\nabla f(x,y,z) = (y, x, 2z) = (0,0,0) \Rightarrow (x,y,z) = (0,0,0)$, and so $f(0,0,0) = 0$. Now, set $g(x,y,z) = x - y$ and $h(x,y,z) = x^2 + y^2 + z^2 - 4$. By Language multipliers, if there are $\mu, \lambda \in \mathbb{R}$ such that

$$\begin{cases} \nabla f(a,b,c) & = \mu \nabla g(a,b,c) + \lambda \nabla h(a,b,c), \\ g(a,b,c) & = h(a,b,c) = 0, \\ \nabla g(a,b,c) & \not\parallel \nabla h(a,b,c) \end{cases} \Leftrightarrow \begin{cases} (b,a,2c) & = (\mu + 2\lambda a, -\mu + 2\lambda b, 2\lambda c), \\ a - b & = a^2 + b^2 + c^2 - 4 = 0, \\ (1,-1,0) & \not\parallel (2a, 2b, 2c) \end{cases}$$

$$\Leftrightarrow \begin{cases} (a,a,2c) & = (\mu + 2\lambda a, -\mu + 2\lambda a, 2\lambda c), \\ 2a^2 + c^2 & = 4, \\ a & = b, \\ (1,-1,0) & \not\parallel (a,b,c) \end{cases} \Leftrightarrow \begin{cases} b = a & = \mu + 2\lambda a = -\mu + 2\lambda a, \\ c & = \lambda c, \\ 2a^2 + c^2 & = 4, \\ (1,-1,0) & \not\parallel (a,b,c) \end{cases}$$

$$\Leftrightarrow \begin{cases} b = a & = 2\lambda a, \\ \mu & = 0, \\ c & = \lambda c, \\ 2a^2 + c^2 & = 4, \\ (1,-1,0) & \not\parallel (a,b,c), \end{cases}$$

then f attains extreme value at (a,b,c) with $g(a,b,c) = 0$ and $h(a,b,c) = 0$.

Case I: $c = 0$. Then $2a^2 = 4$, $a = b = \pm\sqrt{2}$. $f(a,b,c) = 2$.

Case II: $c \neq 0$. Then $\lambda = 1$, $b = a = 2a$, $a = b = 0$. Hence $c = \pm 2$, $f(a,b,c) = 4$.

In summary, the maximum is $f(0,0,0) = 0$, and the minimum is $f(0,0,\pm 2) = 4$.

Ans: (A)(B).

Question 18. [15.4-S-★] The **volume** of the solid lying below the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = z$ is

(A) $\frac{\pi}{24}$. (B) $\frac{\pi}{48}$. (C) $\frac{\pi}{12}$. (D) $\frac{5\pi}{72}$.

Solution 18. [Way I][Double integral in polar coordinate] Notice that

$$\begin{cases} z = \sqrt{x^2 + y^2}, \\ x^2 + y^2 + z^2 = z \end{cases} \Leftrightarrow \begin{cases} z = \sqrt{x^2 + y^2}, \\ 2z^2 = z \end{cases} \Leftrightarrow z = \sqrt{x^2 + y^2} = 0, \frac{1}{2}.$$

Notice that $x^2 + y^2 + z^2 \leq z$ if and only if $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 \leq \left(\frac{1}{2}\right)^2$. Hence the solid is

$$E = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 - (x^2 + y^2)} \leq z \leq \sqrt{x^2 + y^2} \right\}.$$

Remember that $(x_0, y_0, z_0) \in E$ if and only if $(x_0, y_0) \in E_2 := \left\{ (x, y) \,\middle|\, x^2 + y^2 \le \left(\frac{1}{2}\right)^2 \right\}$. By double integral in polar coordinate, the **volume** of the solid E is

$$V = \iint_{(x,y)\in E_2} \sqrt{x^2+y^2} - \left(\frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 - (x^2+y^2)}\right) dA = \int_0^{2\pi}\int_0^{1/2} \left[r - \left(\frac{1}{2} - \sqrt{\frac{1}{4} - r^2}\right)\right] \cdot r\, dr\, d\theta$$

$$= 2\pi \cdot \left[\frac{r^3}{3} - \frac{r^2}{4} - \frac{1}{3}\left(\frac{1}{4} - r^2\right)^{3/2}\Big|_{r=0}^{1/2}\right] = \frac{\pi}{24}.$$

[Way II][Triple integral in spherical coordinate] Notice that .

To convert from rectangular coordinate to spherical coordinate, we have ρ^2

$$x^2 + y^2 + z^2 = z \;\Leftrightarrow\; \rho^2 = \rho\cos\phi \;\Leftrightarrow\; \begin{cases} \rho & \ge 0, \\ \rho & = \cos\phi. \end{cases}$$

Hence the solid is

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 \le z \le \sqrt{x^2+y^2} \right\}$$
$$= \left\{ (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \in \mathbb{R}^3 \,\middle|\, \frac{\pi}{4} \le \phi \le \frac{\pi}{2}, 0 \le \rho \le \cos\phi, 0 \le \theta \le 2\pi \right\}$$

By triple integral in spherical coordinate, the **volume** of the solid E is

$$V = \iiint_{(x,y,z)\in E} dV = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_{\pi/4}^{\pi/2}\left(\int_0^{\cos\phi} \rho^2\, d\rho\right)\sin\phi\, d\phi\right)$$

$$= 2\pi \cdot \left(\int_{\pi/4}^{\pi/2} \frac{\cos^3\phi}{3}\sin\phi\, d\phi\right) = 2\pi \cdot \left(-\frac{\cos^4\phi}{12}\Big|_{\phi=\pi/4}^{\pi/2}\right) = \frac{\pi}{24}.$$

Ans: (A).

Question 19. [15.10-S-★] Let R be the region on the xy-plane bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, $(1, 0)$. If $\iint_R (x + y)^2 \sin^2(x - y)\, dA = \frac{a}{6}(2 - \sin 2)$, then $a =$?

(A) 27. (B) 26. (C) 13. (D) 9.

Solution 19. Let $u = x + y$ and $v = x - y$. Then (x, y) is in the square if and only if $1 \le u \le 3$ and $-1 \le v \le 1$. Notice that the Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}^{-1} = -\frac{1}{2}.$$

By change of variables in a double integral and Fubini's theorem,

$$\iint_R (x + y)^2 \sin^2(x - y)\, dA = \int_{-1}^{1}\int_1^3 u^2 \sin^2 v \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv = \frac{1}{2}\left(\int_1^3 u^2\, du\right)\left(\int_{-1}^{1} \sin^2 v\, dv\right)$$

$$= \frac{1}{2} \cdot \frac{26}{3} \cdot \left(\frac{v}{2} - \frac{\sin 2v}{4}\Big|_{v=-1}^{1}\right) = \frac{13}{3} \cdot \left(1 - \frac{\sin 2}{4} - \frac{\sin(-2)}{4}\right) = \frac{13}{6} \cdot (2 - \sin 2).$$

Ans: (C).

Question 20. [14.4-S-★★★★] Let S be the surface obtained by rotating the curve $y = x^2 + 1$ on the xy-plane about the x-axis. Then, the **tangent plane** to S at $(1, -1, \sqrt{3})$ is

(A) $4x + y - 3z = 0$. (B) $4x + y - \sqrt{3}z = 0$. (C) $4x + y - 3z = 2$. (D) $4x + y - \sqrt{3}z = 2$.

Solution 20. Notice that if point $P = (x, y_1, z_1)$, $Q = (x, y_2, z_2)$ locates on surface S, then the distance between P and x-axis equals to the distance between Q and x-axis. Especially, they should equal to the distance between $R = (x, x^2 + 1, 0)$ and x-axis.

$$\sqrt{(x-x)^2 + (y_i - 0)^2 + (z_i - 0)^2} = x^2 + 1 \;\Leftrightarrow\; y_i^2 + z_i^2 = \left(x^2 + 1\right)^2, \quad i = 1, 2.$$

Let $f(x, y, z) = \left(x^2 + 1\right)^2 - \left(y^2 + z^2\right)$. Then $(x, y, z) \in S$ if and only if $f(x, y, z) = 0$. Now, $\nabla f(x, y, z) = \left(4x(x^2 + 1), -2y, -2z\right)$. Hence the **tangent plane** to S at $(1, -1, \sqrt{3})$ is

$$\begin{aligned} & E:\ \nabla f(1, -1, \sqrt{3}) \cdot \left((x, y, z) - (1, -1, \sqrt{3})\right) = 0 \\ \Leftrightarrow\ & E:\ (8, 2, -2\sqrt{3}) \cdot (x - 1, y + 1, z - \sqrt{3}) = 0 \\ \Leftrightarrow\ & E:\ 4x + y - \sqrt{3}z = 0. \end{aligned}$$

`Ans: (B).`

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