99-2

1.
Using ratio test, we have

$$\lim_{n\to\infty} \frac{\left|\frac{3^{n+1}x^{n+1}}{(n+2)^2}\right|}{\left|\frac{3^n x^n}{(n+1)^2}\right|} = \lim_{n\to\infty} 3\cdot|x|\cdot\frac{(n+1)^2}{(n+2)^2} = 3|x|.$$

The series $\sum_{n=1}^{\infty} \frac{3^n x^n}{(n+1)^2}$ converges, if $3|x|<1$. If $x=\frac{1}{3}$, then the series $\sum_{n=1}^{\infty} \frac{3^n x^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent by p -series test. On the other hand, if $x=-\frac{1}{3}$, then the series $\sum_{n=1}^{\infty} \frac{3^n x^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}$ is convergent, obviously. Hence the interval of convergence is $[-1/3, 1/3]$.
`Ans: B`

2.
Define $a_1 = \frac{1}{4}$, and $a_{n+1} = \frac{1}{4+a_n}$. Note that $a_n > 0$ for all $n \in \mathbb{N}$ and bounded in the interval $[0,1]$. For odd terms, a_n is a decreasing sequence; and for even terms, a_n is an increasing sequence. So a_n will converges to some value, said a. Hence $a = \frac{1}{4+\frac{1}{4+\frac{1}{4+\cdots}}}$, then

$$a = \frac{1}{4+a} \Rightarrow a = -2 \pm \sqrt{5}. \tag{1}$$

Since a must be positive, $a = -2+\sqrt{5}$.
`Ans: C`

3.
Since the Maclaurin series of $\cos x$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

`Ans: B`

4.
Suppose $r(t) = (t^2, t, 1)$, then $r'(t) = (2t, 1, 0)$. Hence $r'(t)\times r(t) = (1, -2t, t^2)$, $r'(t)\cdot r(t) = 2t^3 + t$, and $(r'(t)\times r(t))\times r(t) = (-2t - t^3, -1+t^4, t+2t^3)$. When we take $t_0 = 1$ as an example, then $r'(t)\times r(t) = (1,-2,1)$, $r'(t)\cdot r(t) = 3$, and $(r'(t)\times r(t))\times r(t) = (-3, 0, 3)$. Hence (A),(B), and (C) are incorrect. Suppose $r(t) = (x(t), y(t), z(t))$, then $r'(t) = (x'(t), y'(t), z'(t))$, and $(r'(t)\times r(t))\cdot r(t) = (0,0,0)$.
`Ans: D`

5.
When $(x,y)=(t,t)$, then as $t\to 0$, $(x,y)\to(0,0)$. Hence

$$\lim_{(x,y)\to(0,0)} \frac{x^{\frac{2}{3}}\sin y}{x^2+y^2} = \lim_{t\to 0}\frac{t^{\frac{2}{3}}\sin t}{2t^2}. = \lim_{t\to 0}\frac{t^{\frac{2}{3}}\cdot t}{2t^2}. = \lim_{t\to 0}\frac{t^{-\frac{1}{3}}}{2}.$$

Since $\lim_{t\to 0} t^{-\frac{1}{3}}$ does not exist, the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^{\frac{2}{3}}\sin y}{x^2+y^2}$ does not exist.

`Ans: D`

6.
Doing the partial differential for the equation, we have

$$y\cdot\frac{\partial w}{\partial z} + x^2\cdot 3w^2\frac{\partial w}{\partial z} + 6z + 2zw + z^2\frac{\partial w}{\partial z} - 2y = 0.$$

Substitute $(x,y,z)=(0,1,0)$ into the equation above, we have

$$\frac{\partial w}{\partial z} = 2.$$

`Ans: B`

7.
Since

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y},$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y},$$

$$\frac{\partial z}{\partial s}\frac{\partial z}{\partial t} = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2.$$

`Ans: A`

8.
Note that

$$f_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x},$$

and

$$f_y(x,y) = 2ye^{-x}.$$

$f_x(x,y)=0$ and $f_y(x,y)=0$ if and only if $(x,y)=(0,0)$ or $(2,0)$. Hence $f(x,y)$ has two critical points which are $(0,0)$ and $(2,0)$.
`Ans: C`

9.
Represent the triple integral in spherical coordinate, we have

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^3 e^{-\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Let $u = \rho^3$, then

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^3 e^{-\rho^3}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2}\left(-\frac{1}{3}\int_0^{27} e^{-u}\sin\phi\,du\right)d\phi\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}(-e^{-27}+1)\sin\phi\,d\phi\,d\theta$$

$$= \frac{2\pi}{3}(1-e^{-27}).$$

`Ans: C`

10.
Let $x = r\cos\theta$ and $y = r\sin\theta$, then we have

$$\iint_R \sin(x^2+y^2)dA = \frac{1}{2}\int_1^2\int_0^{2\pi}\sin(r^2)r\,d\theta\,dr$$

$$= \frac{1}{2}\int_1^2 2\pi r\sin(r^2)\,dr.$$

Let $s = r^2$, then

$$\frac{1}{2}\int_1^2 2\pi r\sin(r^2)\,dr = \frac{1}{2}\int_1^4 \pi\sin s\,ds = \frac{\pi}{2}(-\cos 4+\cos 1).$$

`Ans: A`

11.
(A) Since $0 < \left|\dfrac{\sin 4n}{4^n}\right| < \dfrac{1}{4^n}$ and $\sum_{n=1}^{\infty}\dfrac{1}{4^n}$ is convergent, $\sum_{n=1}^{\infty}\dfrac{\sin 4n}{4^n}$ is absolutely convergent by comparison test.
(B) Since $0 \le n+n\cos^2 n < n+n$, $\dfrac{1}{n+n\cos^2 n} > \dfrac{1}{2n} > 0$. And $\sum_{n=1}^{\infty}\dfrac{1}{2n}$ is divergent. Hence $\sum_{n=1}^{\infty}\dfrac{1}{n+n\cos^2 n}$ is divergent.
(C) Since $\dfrac{\ln n}{n}$ is decreasing to 0, by alternative test, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\ln n}{n}$ is convergent.
(D) Since $\sin 1 < 1$, $\dfrac{1}{n^{\sin 1}} > \dfrac{1}{n} > 0$, by comparison test, the series $\sum_{n=1}^{\infty}\dfrac{1}{n^{\sin 1}}$ is divergent.
`Ans: AC`

12.
(A) Take $a_n = \dfrac{1}{n}$ as an example, then $\lim_{n\to\infty}\dfrac{1}{n} = 0$ and $\dfrac{1}{n} > 0$ for all $n \in \mathbb{N}$. But $\sum_{n=1}^{\infty}\dfrac{1}{n}$ is divergent.
(B) Since $\sum_{n=1}^{\infty} a_n$ is convergent, $\lim_{n\to\infty} a_n = 0$. Then

$$\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty}|a_n| = 0.$$

(C)
By **James Stewart, Calculus Early Transcendentals, 7th ed., pp.737**.

(D) The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$. But $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if and only if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent. So $\sum_{n=1}^{\infty} a_n$ is absolutely convergent doesn't mean $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

`Ans: BC`

13.

Note that

$$f_x(x,y) = (2-2x)(2y-y^2),$$
$$f_y(x,y) = (2x-x^2)(2-2y),$$
$$f_{xx} = -2(2y-y^2),$$
$$f_{yy} = (2x-x^2)(-2),$$

and

$$f_{xy} = (2-2x)(2-2y).$$

$f_x(x,y) = 0$ and $f_y(x,y) = 0$ if and only if $(x,y) = (1,1)$, $(2,2)$, $(2,0)$, $(0,2)$, or $(0,0)$. Since

$$f_{xx}(0,0) \cdot f_{yy}(0,0) - f_{xy}(0,0) \cdot f_{yx}(0,0) = -16 < 0,$$

the critical point $(0,0)$ is a saddle point.

$$f_{xx}(2,0) \cdot f_{yy}(2,0) - f_{xy}(2,0) \cdot f_{yx}(2,0) = -16 < 0,$$

the critical point $(2,0)$ is a saddle point. Hence $f(x,y)$ has more than one saddle point. On the other hand,

$$f_{xx}(1,1) \cdot f_{yy}(1,1) - f_{xy}(1,1) \cdot f_{yx}(1,1) = 4 > 0,$$

and

$$f_{xx}(1,1) = -2 < 0,$$

so the critical point $(1,1)$ is a local maximum and $f(1,1) = 1$. Note that

$$f(3,3) = (-3) \cdot (-3) = 9 > 1 = f(1,1).$$

Hence $(1,1)$ is not an absolute maximum.

`Ans: ABC`

14.

(A) When $(x,y) = (t,0)$, then as $(x,y) \to (0,0)$, $t \to 0$. Hence

$$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{x^2+y^2} = \lim_{t\to 0} \frac{-t^2}{t^2} = -1 \neq 1.$$

So f is not continuous at $(0,0)$.

(B) Since f is a rational function, and the denominator of f are not 0, when $(x,y) \neq (0,0)$, f is differentiable at $(0,y)$ for $y \neq 0$.

(C)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{-1-1}{h}$$

doesn't exist.
(D) Since

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0,$$

$f_y(0,0)$ exists.
`Ans: BD`

15.
The range is $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, and $0 \le z \le x$. (A) (B): Since the bound of y and z are only depend on x, we can change the integral directly.
(C) (D): Since $0 \le x \le 1$ and $0 \le z \le x$, we have $0 \le z \le x \le 1$. And since $0 \le y \le \sqrt{1-x^2}$ and $z \le x$, we have $0 \le y \le \sqrt{1-z^2}$. Finally, from $z \le x$ and $y \le \sqrt{1-x^2}$, we have $z \le x \le \sqrt{1-y^2}$.
`Ans: BD`

16.
Note that $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Hence

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1},$$

and

$$\int_0^x \frac{\tan^{-1} t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}.$$

`Ans:` $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$

17.
Let $g(x,y,z) = y - x$, and $h(x,y,z) = x^2 + y^2 + z^2 - 4$. Using Lagrange multiplier, we should solve the equation for x, y and z

$$\nabla f + \alpha \nabla g + \beta \nabla h = (0,0,0)$$

$$\Rightarrow (y, x, 2z) + (-\alpha, \alpha, 0) + (2\beta x, 2\beta y, 2\beta z) = (0,0,0).$$

Next, we solve the system of equations

$$\begin{cases} y - \alpha + 2\beta x = 0 \\ x + \alpha + 2\beta y = 0 \\ 2z(1+\beta) = 0. \end{cases}$$

Then, we have four critical points which are $(x,y,z) = (0,0,2)$, $(0,0,-2)$, $(\sqrt{2}, \sqrt{2}, 0)$ and $(\sqrt{2}, \sqrt{2}, 0)$. By putting the values, the maximum is 4 while $(x,y,z) = (0,0,\pm 2)$.
`Ans: 4`

18.
Change the arrangement of the the integral, then we have

$$\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x} dx dy = \int_0^{\pi} \int_0^{x} \frac{\sin x}{x} dy dx$$

$$= \int_0^{\pi} \sin x dx = 2$$

`Ans:` 2

19.
The region is bounded by $y - x = 0$, $y - x = 2$, $x + y = 2$, and $x + y = 4$. Let $u = y - x$ and $v = y + x$, then the Jacobian matrix is

$$J = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Hence

$$\iint_R \frac{y - x}{y + x} dA = \int_0^2 \int_2^4 \frac{u}{v} \cdot \left| -\frac{1}{2} \right| dv du = \ln 2.$$

`Ans:` $\ln 2$

20.
We divide the solid into three parts, which are two cones and the remaining part. Note that the volume of a cone is $\frac{1}{3} \cdot (\text{Area of buttom}) \cdot (\text{Height})$. Hence we can write down the volume as follows

$$V = \int_0^{1/2} \int_0^{2\pi/3} \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho + \frac{1}{3} \cdot \left(\frac{\sqrt{3}}{4} \right)^2 \cdot \pi \cdot \frac{1}{4} + \frac{1}{3} \cdot \left(\frac{\sqrt{3}}{4} \right)^2 \cdot \pi \cdot \frac{1}{4}$$

$$= \frac{\pi}{8} + \frac{\pi}{64} + \frac{\pi}{64} = \frac{5\pi}{32}.$$

`Ans:` $\frac{5\pi}{32}$